# XVII. *On a New Geometry of Space.* *By* J. Plücker, *of Bonn, For. Memb. R.S.*

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## I. *On Linear Complexes of Right Lines.*

1. Infinite space may be considered either as consisting of points or transversed by planes. The points, in the first conception, are determined by their coordinates, by $x$, $y$, $z$ for instance, taken in the ordinary signification; the planes, in the second conception, are determined in an analogous way by their coordinates, introduced by myself into analytical geometry, by $t$, $u$, $v$ for instance.

The equation

$$tx+uy+vz+1=0$$

represents, in regarding $x$, $y$, $z$ as variable and $t$, $u$, $v$ as constant, a plane by means of its points. The three constants $t$, $u$, $v$ are the coordinates of this plane. The same equation, in regarding $t$, $u$, $v$ as variable, $x$, $y$, $z$ as constant, represents a point by means of planes passing through it. The three constants are the coordinates of the point.

A point given by its coordinates and a point determined by its equation, or geometrically speaking by an infinite number of planes intersecting each other in that point, are quite different ideas, not to be confounded with one another. That is the case also with regard to a plane given by its coordinates and a plane represented by its equation, or considered as containing an infinite number of points. Hence is derived a double signification of a right line. It may be considered as the geometrical locus of points, or described by a point moving along it, and accordingly represented by two equations in $x$, $y$, $z$, each representing a plane containing that line. But it may likewise be considered as the intersection of an infinite number of planes, or as enveloped by one of these planes, turning round it like an axis; accordingly it is represented by two equations in $t$, $u$, $v$, each representing an arbitrary point of the line. The passage from one of the two conceptions to the other is a discontinuous one*.

2. The geometrical constitution of space, hitherto referred either to points or to planes, may as well be referred to right lines. According to the double definition of such lines, there occurs to us a double construction of space.

In the first construction we imagine infinite space to be transversed by lines themselves consisting of points. An infinite number of such lines pass in all directions through any given point; each of these lines may be regarded as described by a moving

* According to this discontinuity, a plane curve represented by ordinary coordinates may have a conjugate which disappears if the same curve be represented by means of line-coordinates. See "System der analytischen Geometrie," n. 330.

point. This constitution of space is admitted when, in optics, we consider luminous points as sending out in all directions luminous rays, or, in mechanics, forces acting on points in every direction.

In the second construction infinite space is likewise regarded as transversed by right lines, but these lines are determined by means of planes passing through them. Every plane contains an infinite number of right lines having within it every position and direction, around each of which the plane may turn. We refer to this second conception when, in optics, we regard, instead of rays, the corresponding fronts of waves and their consecutive intersections, or when, in mechanics, according to POINSOT's ingenious philosophical views, we introduce into its fundamental principles "couples," as well entitled to occupy their place as ordinary forces. The instantaneous axes of rotation are right lines of the second description.

3. In order to constitute a new geometry of space, we may fix the position of a right line, depending upon four constants, in a different way. We might do it by means of four given right lines, by determining, for instance, the shortest distance of any new line from each of the four given ones. But all such conceptions were rejected, and the ordinary system of axes adopted in order to fix the position in space of a right line. Thus the new researches, indicated by the foregoing remarks, are intimately connected with the usual methods of analytical geometry. The two fragments presented on this occasion are only calculated to give an exact idea of the new way of proceeding, and to show its importance, greater perhaps than it appears at first sight.

4. A right line of the first description, which we shall distinguish by the name of *ray*, may be determined by means of two of its projections. We may select the projections within the planes XZ and YZ, in order to get, without generalizing, the greatest symmetry obtainable, and give to their equations either the form

$$\left.\begin{aligned} x&=rz+\varrho, \\ y&=sz+\sigma, \end{aligned}\right\} \qquad (1)$$

or

$$\left.\begin{aligned} tx+v_x z&=1, \\ uy+v_y z&=1. \end{aligned}\right\} \qquad (2)$$

In adopting the first system of equations, the four constants $r$, $s$, $\varrho$, $\sigma$ are *the coordinates of the ray*: two of them, $r$, $s$, indicating its direction, the remaining two, $\varrho$, $\sigma$, after its direction is determined, giving its position in space. The ray meets the plane XY in the point

$$x=\varrho, \qquad y=\sigma.$$

In adopting the second system of equations, we get, in order to determine the same ray, the four new constants $t$, $u$, $v_x$, $v_y$, which likewise may be regarded as its coordinates; $t$ and $u$ $\left(\text{equal to } \frac{1}{\varrho} \text{ and } \frac{1}{\sigma}\right)$ indicating the reciprocal values of the intercepts cut off on OX and OY by the two projections of the ray, $v_x$ and $v_y$ $\left(\text{equal to } \left(-\frac{r}{\varrho}\right)\right.$ and $\left.\left(-\frac{s}{\sigma}\right)\right)$ the reciprocal values of the two intercepts cut off both on OZ.

5. A right line of the second description, which we shall distinguish by the name of *axis*, is determined by any two of its points. We may select the intersection of the axis with the planes XZ and YZ as two such points, and represent them by the system of equations

$$\left.\begin{aligned} xt+z_t v&=1, \\ yu+z_u v&=1, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (3)$$

or by the following equally symmetrical,

$$\left.\begin{aligned} t&=pv+\varpi, \\ u&=qv+\varkappa. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4)$$

In making use of the first two equations, the four constants $x$, $y$, $z_t$, $z_u$ are *the coordinates of the axis*, indicating the position of the two points within the planes XZ, YZ.

In making use of the second system of equations, $p$, $q$, $\varpi$, $\varkappa$ are the four coordinates of the axis, this axis being fixed by the intersections of two planes, one of which is the plane projecting it on XY, and determined by two of the four coordinates,

$$t=\varpi=\frac{1}{x}, \quad u=\varkappa=\frac{1}{y},$$

while the other plane determined by the two remaining ones,

$$t=pv=-\frac{z_t}{x}v, \quad u=qv=-\frac{z_u}{y}v,$$

and represented by the equation

$$px+qy+z=0,$$

passes through the axis and the origin.

6. If we consider the four coordinates of a ray as variable quantities, we may in attributing to them any given values successively obtain any ray whatever transversing space. But in admitting that an equation takes place between the four coordinates, rays are excluded: we say *that the remaining rays constitute a complex represented by the equation.*

In admitting *two* such equations existing simultaneously, those rays the coordinates of which satisfy both equations constitute a *congruency represented by the system of equations.* A "congruency" contains all congruent rays of two complexes, it may be regarded as their mutual intersection. If we admit that three equations are simultaneously verified by the four coordinates, the corresponding rays constitute a *configuration* (Strahlengebilde, surface reglée) *represented by the system of three equations.* A configuration may be regarded as the mutual intersection of three complexes, *i. e.* as the geometrical locus of congruent rays belonging to all three complexes. Four complexes or two configurations intersect each other in a limited number of rays. The number of rays constituting a configuration, a congruency, a complex, and space, are infinites of first, second, third, and fourth order.

7. If *rays* are replaced by axes, complexes, congruencies, and configurations of rays are replaced by *complexes, congruencies, and configurations of axes.*

8. A configuration of rays or axes, represented by three linear equations, is, according to the choice of coordinates, either a hyperboloid or a paraboloid. Let the three equations of a configuration of rays be

$$\left.\begin{aligned} Ar+Bs+C+D\sigma+E\varrho&=0,\\ A'r+B's+C'+D'\sigma+E'\varrho&=0,\\ A''r+B''s+C''+D''\sigma+E''\varrho&=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From these equations we derive by elimination six new ones, each containing two only of the four variables. Let them be

$$ar+bs=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$c\varrho+d\sigma=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$a'r+c'\varrho=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$b's+d'\sigma=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$a''r+d''\sigma=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$b''s+c''\varrho=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In order to represent the configuration, the three primitive equations (5) may be replaced by any three of the six new ones.

The equation (7) may be written thus,

$$cx+dy=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^*)$$

$x$ and $y$ replacing $\varrho$ and $\sigma$. It represents a right line within XY, intersected by the rays of the configuration.

The equations (8) and (9) represent within XZ, YZ two points enveloped by the projections of the rays of the configuration; consequently the rays themselves meet two right lines passing through these points, and being parallel to OY, OZ. From the equations (8) and (9) if written thus,

$$\frac{1}{c'}=\frac{a'}{c'}.r+\varrho,$$

$$\frac{1}{d'}=\frac{b'}{d'}.s+\sigma,$$

we immediately derive

$$c'x=1, \quad c'z=a',$$

$$d'y=1, \quad d'z=b',$$

representing the two right lines.

Thus by selecting in order to represent the configuration the three equations (7), (8), (9), and interpreting them geometrically, we have proved that all its rays intersect three fixed right lines, one of which falls within XY, while the two remaining ones are parallel to OY and OX. Hence these rays, meeting three right lines parallel to the same plane, constitute a hyperbolic paraboloid.

In determining the paraboloid, we may replace any one of the three equations we

made use of by the equation (6), which indicates that all rays are parallel to a given plane. This plane, if drawn through the origin, is represented by the equation

$$ax+by=z,$$

obtained from (6) by writing $\frac{x}{z}$, $\frac{y}{z}$ instead of $r$, $s$.

It may be sufficient here to state that a configuration of rays, if represented by three linear equations, in which the coordinates $r$, $s$, $\varrho$, $\sigma$ are replaced by $t$, $u$, $v_x$, $v_y$, becomes a hyperboloid.

9. A configuration of *axes* represented by three linear equations would be a *paraboloid* if the coordinates $x$, $y$, $z_t$, $z_u$ were employed, but becomes a *hyperboloid* if these coordinates are replaced by $p$, $q$, $\varpi$, $\varkappa$. We shall here consider the last case only, and may for that purpose directly replace the equations (6)–(11) by the following ones:—

$$ap+bq=1, \quad (12)$$

$$c\varpi+d\varkappa=1, \quad (13)$$

$$a'p+c'\varpi=1, \quad (14)$$

$$b'q+d'\varkappa=1, \quad (15)$$

$$a''p+d''\varkappa=1, \quad (16)$$

$$b''q+c''\varpi=1. \quad (17)$$

Any three of these equations, involving six constants, are sufficient to determine the configuration.

If, after having replaced $p$, $q$, $\varpi$, $\varkappa$ by

$$-\frac{z_t}{x}, \quad -\frac{z_u}{y}, \quad \frac{1}{x}, \quad \frac{1}{y},$$

we regard $x$, $y$, $z_t$, $z_u$ as variable, (14) and (15) may be written thus,

$$x=a'z+c',$$

$$y=b'z+d',$$

representing within the planes XZ, YZ two right lines (AA$'$, BB$'$) which are the locus of points (A, B) where the axes of the configuration meet the two planes.

In regarding $\varpi$ and $\varkappa$ as coordinates of a right line, the equation (13), being written thus,

$$ct+du=1,$$

represents a given point (E),

$$x=c, \quad y=d,$$

enveloped within XY by the projections of axes. Therefore all axes of the configuration intersect a third right line (CC$'$) parallel to OZ and meeting XY in E.

Hence we conclude that the configuration represented by the three linear equations is a hyperboloid. Its axes meet three given lines, two of which, AA$'$, BB$'$, fall within XZ, YZ, while the third, CC$'$, is parallel to OZ.

The plane BOA passing through O and an axis AB is represented by the equation

$$z+qy+px=0.$$

The equation (12) being with regard to $p$ and $q$ of the first degree, indicates that all such planes, containing the different axes of the configuration, intersect each other along a given right line DD$'$ passing through O. Hence all axes meet a fourth right line, itself confined within the hyperboloid.

The complete determination of the hyperboloid presents no difficulties. We may for instance find its centre and its axes by determining the shortest distance of any two of the axes generating it.

10. Let a *congruency* either of rays or axes be represented by two linear equations. In adding to these equations two new ones, likewise of the first degree, there exists only one ray or axis the coordinates of which satisfy simultaneously the four linear equations. Two new equations of this description are obtained if, among the rays or axes of the congruency, we select those either passing through a given point, or confined within a given plane. In the case of rays, let $(x', y', z')$ be a given point, then we get

$$x'=rz'+\varrho,$$
$$y'=sz'+\sigma$$

in order to express that all rays meet in that point. Let

$$t'x+u'y+v'z+1=0$$

be the equation of a given plane, then we get

$$t'r+u's+v=0,$$
$$t'\varrho+u'\sigma+1=0$$

in order to express that the rays lie within that plane. Again, in the case of axes, let $(t', u', v')$ be a given plane, then we get the new linear equations

$$t'x+v'z_t=1, \qquad t'=pv'+\varpi,$$

or

$$u'x+v'z_u=1, \qquad u'=qv'+\varkappa,$$

in order to express that the axis is confined within that plane. Let in regarding $x', y', z'$ as constant, $t$, $u$, $v$ as variable,

$$x't+y'u+z'v+1=0$$

represent a given point, then we get

$$x'p+y'q+z'=0,$$
$$x'\varpi+y'\varkappa+1=0$$

in order to express that the axes pass through that point. Hence

*In a congruency represented by the system of two linear equations, there is one single ray or axis passing through any given point of space, as there is one single ray or axis confined within a given plane.*

11. In order to represent a congruency of rays, we shall here make use of the coordinates $t$, $u$, $v_x$, $v_y$. Let

$$At+Bu+Cv_x+Dv_y+1=0,$$
$$A't+B'u+C'v_x+D'v_y+1=0$$

be its two equations. By successively eliminating each coordinate, we get four equations of the following form,

$$at+bu+cv_x+1=0,$$
$$a't+b'u+dv_y+1=0,$$
$$a''t+c'v_x+d'v_y+1=0,$$
$$b''u+c''v_x+d''v_y+1=0,$$

any two of which involving six constants may replace the two primitive equations, the remaining two being derived from them.

The first two of these equations, if $t$, $u$, $v_x$ and $t$, $u$, $v_y$ be considered as plane coordinates, represent two points (U, V) the coordinates of which are

$$x=a, \qquad y=b, \qquad z=c, \quad \ldots\ldots\ldots\ldots \quad (\mathrm{U})$$
$$x=a', \qquad y=b', \qquad z=c', \quad \ldots\ldots\ldots\ldots \quad (\mathrm{V})$$

Consequently the six constants upon which the congruency depends, if referred to the three axes of coordinates OX, OY, OZ, are determined by means of the two points U and V. Hence is derived the following construction of rays of the congruency.

Trace through the two points U, V any two planes which intersect each other along a right line confined in the plane XY, and meeting OX, OY in the points D, F. Let E, G be the points where the two planes meet OZ. We shall get within the planes XZ, YZ the projections of a ray of the congruency by drawing DE, FG. The ray (AC) thereby completely determined will intersect the plane XY in the point C, the coordinates of which are

$$x=\frac{1}{t}=\mathrm{OD}, \qquad y=\frac{1}{u}=\mathrm{OF}.$$

If a plane be traced passing simultaneously through both points U, V, both intersections E, G falling into one point A', the corresponding ray of the congruency A'C' intersects OZ. If the right line UV be projected on YZ, XZ, the projections meet OZ in two points A'', A'''. In these points OZ is intersected by the rays of the congruency parallel to OX, OY. The ray parallel to OZ is obtained by the point C'' where it meets XY. The coordinates of C'' are

$$x=\mathrm{OD}'', \qquad \mathrm{Y}=\mathrm{OF}'',$$

D'' and F'' being the points where the projection of UV intersects OX and OY.

Thus occurs to us the construction of rays passing through any point of OZ and any point of XY. We cannot go further into detail here.

12. Again, let a congruency of *axes* be represented by the equations

$$Ax+By+Cz_t+Dz_u+1=0,$$
$$A'x+B'y+C'z_t+D'z_u+1=0.$$

By successively eliminating $z_u$ and $z_t$ we may replace these equations by the following two,

$$ax+by+cz_t+1=0,$$
$$a'x+b'y+c'z_u+1=0,$$

the six new constants of which are derived from the primitive constants. In regarding $x$, $y$, $z_t$, $z_u$ as point-coordinates (where $z$ may be written instead of $z_t$ and $z_u$), the last equations represent two planes. The six coordinates of both planes,

$$t=a, \qquad u=b, \qquad v=c,$$
$$t=a', \qquad u=b', \qquad v=c',$$

are the six constants of the congruency, consequently the congruency is determined by means of these two planes and the axes of coordinates.

Suppose both planes to be known. Draw any right line meeting them in M and M′, project M on XZ and M′ on YZ. The right line joining the two projections B and A is an axis of the congruency.

If we project on XZ and YZ any point of the right line JK along which both planes intersect each other, the right line joining both projections, B′, A′, is an axis parallel to XY. All axes obtained in that way meet, within XZ and YZ, both projections of JK. Hence the axes of the congruency parallel to XY constitute a paraboloid. The ray within XY is obtained by projecting the point where the traces of both planes meet on OX and OY and joining both projections, B″ and A″, by a right line, &c.

13. After these preliminary discussions we shall now proceed in a more systematic way, and henceforth exclusively make use of the coordinates $r$, $s$, $\varrho$, $\sigma$. When a complex of rays is represented by the linear equation

$$Ar+Bs+D\sigma+E\varrho+1=0, \quad \ldots \ldots \quad (1)$$

we may easily prove that the infinite number of rays passing through a given point of space are confined within the same plane, and, conversely, that the infinite number of rays confined within a given plane meet within the same point.

In order to select among the rays of the complex those passing through a given point $(x', y', z')$, the following two equations,

$$\left.\begin{aligned} x'&=rz'+\varrho, \\ y'&=sz'+\sigma, \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (2)$$

are to be added to the equation of the complex. By eliminating $\varrho$ and $\sigma$ we get

$$(A-Ez')r+(B-DZ')s+(1+Ex'+Dy')=0. \quad \ldots \ldots \quad (3)$$

This equation being of the first degree with regard to the remaining variables $r$ and $s$, shows that all corresponding rays are parallel to a given plane, and therefore confined

within the plane of that direction and passing through the point $(x', y', z')$. By replacing in the last equation $r$ and $s$ by $\frac{x-x'}{z-z'}$, and $\frac{y-y'}{z-z'}$, we obtain, in order to represent that plane, the following equation,

$$(A-Ez')(x-x')+(B-Dz')(y-y')+(1+Ex'+Dy')(z-z')=0. \quad . \quad . \quad . \quad (4)$$

14. Again, this equation being, with regard to $(x', y', z')$, of the first degree, proves that, conversely, all rays confined within a given plane meet in the same point of that plane.

15. A complex the rays of which are distributed through infinite space in such a way that in each point there meet an infinite number of rays constituting a plane, and, conversely, that each plane contains an infinite number of rays meeting in the same point, may be called a *linear complex of rays*. We may say, too, that, with regard to the complex, points and planes of the infinite space *correspond to each other*; each plane containing all rays which meet in the point placed within it, and each point being traversed by all rays which are confined within the plane passing through it.

16. A linear complex of rays is represented by the linear equation (1), but it is easily seen that this equation is not the *general* equation of a linear complex. The following considerations lead us to generalize the preceding developments and to render them by generalizing more symmetrical.

Hitherto we determined a ray by its two projections within XZ, YZ,

$$x=rz+\varrho,$$
$$y=sz+\sigma,$$

whence its third projection within XY is derived,

$$ry-sx=r\sigma-s\varrho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This equation furnishes the new term $(r\sigma-s\varrho)$, which, like $\varrho$ and $\sigma$, depend upon $r$ and $s$ as well as upon $x'$ and $y'$ in a linear way.

Again, from the equations

$$tr+us+v=0,$$
$$t\varrho+u\sigma+w=0,$$

expressing that the ray $(r, s, \varrho, \sigma)$ falls within the plane $(t, u, v, w)$ represented by the equation

$$tx+uy+vz+w=0\text{*},$$

we deduce

$$\frac{w}{t}\cdot s-\frac{v}{t}\sigma=(r\sigma-s\varrho). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

* Henceforth we shall make use of four plane-coordinates $t$, $u$, $v$, $w$, and accordingly represent a point by a homogeneous equation. Sometimes, where symmetry and brevity require it, likewise $x$, $y$, $z$ shall be replaced by $\xi/\theta$, $\eta/\theta$, $\zeta/\theta$. Accordingly, by introducing the four point-coordinates $\xi$, $\eta$, $\zeta$, $\theta$, a plane is represented by a homogeneous equation.

17. After introducing a new term containing $(s\varrho-r\sigma)$, the equation of the complex may be written thus,

$$Ar+Bs+C+D\sigma+E\varrho+F(s\varrho-r\sigma)=0. \quad \ldots \quad (7)$$

When, after $(r\sigma-s\varrho)$ is eliminated by means of the equation

$$ry'-sx'=r\sigma-s\varrho,$$

we proceed as we did in the former case [14], the following equation is obtained in order to represent the plane corresponding to the given point $(x', y', z')$,

$$(A-Fy'-Ez')(x-x')+(B+Fx'-Dz')(y-y')+(C+Ex'+Dy')(z-z')=0. \quad . \quad (8)$$

This equation may be expanded thus,

$$(A-Fy'-Ez')x+(B+Fx'-Dz')y+(C+Ex'+Dy')z=Ax'+By'+Cz', \quad . \quad . \quad (9)$$

and reduced also to the following symmetrical form,

$$A(x-x')+B(y-y')+C(z-z')+D(y'z-z'y)+E(x'z-z'x)+F(x'y-y'x)=0. \quad (10)$$

18. We may directly prove that all rays confined within a given plane meet in the same point. The equation of this plane being

$$t'x+u'y+v'z+w'=0, \quad \ldots \quad (11)$$

we get, in order to express that a ray falls within that plane, the following three equations,

$$\begin{aligned} t'r+u's+v' &=0, \\ t'\varrho+u'\sigma+w' &=0, \\ w's-v'\sigma-(r\sigma-s\varrho)t' &=0, \end{aligned}$$

each of which results from the other two. Between these equations and the equation of the complex $(r\sigma-s\varrho)$, $r$ and $\varrho$ may be eliminated. The resulting equation,

$$(Bt'-Au'-Fw')s+(Dt'-Eu'+Fv')\sigma+Ct'-Av'-Ew'=0, \quad . \quad . \quad . \quad (12)$$

being linear with regard to the two remaining variables $s$ and $\sigma$, represents a right line parallel to OX and intersecting YZ in a point, the coordinates of which are

$$\left.\begin{aligned} z' &= \frac{Bt'-Au'-Fw'}{Dt'-Eu'-Fv'}, \\ y' &= -\frac{Ct'-Av'-Ew'}{Dt'-Eu'+Fv'}. \end{aligned}\right\} \quad \ldots \quad (13)$$

Hence all rays of the complex supposed to fall within the plane (11) intersect that right line, and consequently meet in the same point. Two coordinates of that point are given by the last equations, the third,

$$\left. x'=\frac{Cu'-Bv'-Dw'}{Dt'-Eu'+Fv'}, \right\} \quad \ldots \quad (14)$$

is obtained by introducing the values of $z'$ and $y'$ into the equation of the plane.

We may represent the point corresponding to the given plane $(t', u', v', w')$ by its equation,

$$(Cu'-Bv'-Dw')t-(Ct'-Av'-Ew')u+(Bt'-Au'-Fw')v+(Dt'-Eu'+Fv')w=0, \quad (15)$$

which may be written thus,

$$A(v'u-u'v)+B(t'v-v't)+C(u't-t'u)+D(t'w-w't)+E(w'u-u'w)+F(v'w-w'v)=0. \quad (16)$$

19. It is easily seen that both equations (12) and (16) are the most general ones, indicating the supposed correspondence between point and plane. Therefore (10) is the most general equation of a *linear complex*.

20. According to the fundamental relation which characterizes a linear complex, the plane corresponding to a given point is determined by means of any two rays passing through that point, as the point corresponding to a given plane is determined by any two rays confined within that plane.

Suppose P and P′ to be any two points of space, and $p$ and $p'$ the two corresponding planes. Let I be the right line joining both points, II the right line along which both planes intersect each other. Draw through I any plane intersecting II in Q, join Q to P and P′ by two right lines QP, QP′. These right lines, both passing through points (P, P′) and falling within planes $(p, p')$ which pass through them, are rays of the complex. The plane PQP′, containing both rays and consequently containing I, corresponds to the point Q, whence we conclude that planes passing through any points Q, Q′ of II intersect each other along I. Likewise it may be proved that any plane drawn through II intersects I in the corresponding point. We shall call I and II *two right lines conjugate with regard to the linear complex*, or merely *conjugate lines*. The relation between two conjugate lines is a reciprocal one; each of them may be regarded as an axis in space around which a plane turns while the corresponding point describes the other; each also may be regarded as a ray, described by a moving point, the corresponding plane of which turns around the other.

*Each right line meeting two conjugate right lines is a ray of the comp*

To each right line of space there is a conjugate one.

If a point move along a *ray of the complex*, the corresponding plane—containing each ray of the complex which passes through the point, and therefore especially the given one—turns around the ray.

Each ray of the complex may be regarded as two *coincident* conjugate lines.

21. We may also connect the preceding results with the general *principle of polar reciprocity*. Indeed the general equation (10), which represents the plane corresponding to a given point, is not altered if $x', y', z'$ and $x, y, z$ be replaced by one another. Consequently we may say, in introducing the denominations pole and polar plane instead of corresponding point and plane, that the polar planes of all points of a given plane pass through its pole, and conversely, that the poles of all planes passing through a given point fall within the polar plane of that point. In our particular case a plane,

containing its own pole, is determined by means of the poles of any two planes passing through that pole; likewise a point, falling within its polar plane, is determined by means of the polar planes of any two points of its polar plane. A right line joining any two points of space is *conjugate* to the right line, along which the polar planes of both points intersect each other. If one of two conjugate right lines envelopes within a given plane a curve, the other describes a conical surface; the vertex of the cone falls within the plane containing the enveloped curve. Generally if one of the two conjugate right lines describes a configuration, the other one likewise describes such a surface. If one of the two surfaces degenerates into a cone, the other degenerates into a plane curve*.

22. *A point of space being given, to construct the plane which contains all rays of the complex passing through the point.*

Each ray intersecting two conjugate lines is a ray of the complex. Accordingly the only right line starting from a given point and meeting any two conjugate is a ray of the complex. We obtain a new ray, starting from the same point, by means of each new pair of conjugate lines. All such lines constituting the plane corresponding to the given point, two pairs of conjugate lines are sufficient to determine that plane.

*A plane of space being given, to construct the point where meet all rays of the complex confined within the plane.*

Each right line joining the two points in which two conjugate right lines are intersected by a given plane being a ray of the complex, there will be obtained, within the given plane, as many rays as there are known pairs of conjugate lines. Any two such pairs are sufficient in order to determine the point within the plane corresponding to it where all rays meet.

A plane is intersected by the two lines of each conjugate pair in two points; the right lines joining two such points are rays of the complex converging all towards the point which corresponds to the plane. Again, the two planes passing through a point of space and meeting the two lines of a conjugate pair, intersect each other along a ray of the complex confined within the plane which corresponds to the point.

23. After this geometrical digression, immediately indicated by analysis, we resume the analytical way.

By putting in the general equation (9) of the plane corresponding to a given point $(x', y', z')$,

$$x'=0, \quad y'=0, \quad z'=0,$$

we obtain

$$\mathrm{A}x+\mathrm{B}y+\mathrm{C}z=0, \quad \ldots \ldots \ldots \ldots \quad (17)$$

in order to represent the plane corresponding to the origin.

* The peculiar kind of polar reciprocity we meet here was first noticed by M. Möbius in the 10th volume of 'Crelle's Journal,' and was afterwards expounded by L. F. Magnus in his valuable work 'Sammlung von Aufgaben und Lehrsätzen aus der analytischen Geometrie des Raumes,' pp. 139–145.

By putting successively

$$z'=\infty,$$
$$y'=\infty,$$
$$x'=\infty,$$

the same equation becomes

$$\left.\begin{aligned} C+Ex+Dy&=0,\\ B+Fx-Dz&=0,\\ A-Fy-Ez&=0.\end{aligned}\right\} \quad \ldots \quad (18)$$

Accordingly these equations represent the planes corresponding to points moved to an infinite distance along OZ, OY, OX.

By combining each of the equations (18) with (17), we get the rays conjugate to the axes of coordinates OZ, OY, OX, forming a triangle, the angles of which fall within the three planes of coordinates, XY, XZ, YZ, into the corresponding points.

24. By putting

$$w'=\infty,$$

the equation (15), representing a point corresponding to any given point $(x', y' z')$, becomes

$$Dt+Eu-Fv=0,$$

and then indicates that the point corresponding to the infinitely distant plane of space falls itself, at an infinite distance, along a direction which may be represented by the equations

$$\frac{x}{D}=\frac{y}{E}=\frac{z}{F}, \quad \ldots \quad (19)$$

while, if rectangular coordinates were supposed,

$$Dx+Ey+Fz=0$$

represents the plane perpendicular to it.

We shall call this direction *the characteristic direction of the complex.* It is invariably connected with the complex.

25. By putting successively

$$t'=\infty,$$
$$u'=\infty,$$
$$v'=\infty,$$

we get, in order to represent within the planes of coordinates YZ, XZ, XY, the points corresponding to these planes, the following equations:

$$\left.\begin{aligned} Cu-Bv-Dw&=0,\\ Ct-Av-Ew&=0,\\ Bt-Au-Fw&=0.\end{aligned}\right\} \quad \ldots \quad (20)$$

Accordingly the coordinates of these points are

$$\left.\begin{array}{lll} x=0, & y=-\frac{C}{D}\equiv y_t, & z=\frac{B}{D}\equiv z_t, \\ y=0, & x=-\frac{C}{E}\equiv x_u, & z=\frac{A}{E}\equiv z_u, \\ z=0, & x=-\frac{B}{F}\equiv x_v, & y=\frac{A}{F}\equiv y_v, \end{array}\right\} \quad \ldots \ldots \quad (21)$$

whence may be derived the following relation,

$$\frac{x_v y_t z_u}{x_u y_v z_t}=-1.$$

In putting $C=-1$, the right line conjugate to OZ, if regarded as an axis, may be determined by its four coordinates [5],

$$p=A, \quad q=B, \quad \varpi=D, \quad \varkappa=E.$$

These coordinates therefore are four of the constants of the complex

$$Ar+Bs+D\sigma+E\varrho+F(s\varrho-s\sigma)=1.$$

MN conjugate to OZ remains the same whatever may be the value of F. If by putting F equal to zero the last equation becomes a linear one, the complex is completely determined by MN conjugate to OZ.

26. The ratio of the three constants upon which the characteristic direction of the linear complex (1) depends,

$$D : E : F,$$

remains the same if the origin be changed or the complex moved parallel to itself. But if by turning the complex the characteristic direction simultaneously move, that ratio is altered. One of the three constants F, E, D becomes zero if the characteristic direction be confined within XY, XZ, YZ; two of them disappear, F and E, F and D; E and D if that direction fall within OX, OY, OZ. Here the general equation becomes

$$\left.\begin{array}{l} Ar+Bs+C+D\sigma =0, \\ Ar+Bs+C+E\varrho =0, \\ Ar+Bs+C+F(s\varrho-r\sigma)=0. \end{array}\right\} \quad \ldots \ldots \quad (22)$$

27. The ratio of the three constants

$$A : B : C$$

varies if the complex be moved parallel to itself. If the plane corresponding to O pass through OZ, OY, OX, one of the three constants C, B, A becomes zero; if this plane be congruent with XY, XZ, YZ, *i. e.* if O be the point corresponding to XY, XZ, YZ, two constants A and B, A and C, B and C disappear, and the general equation of the

complex becomes

$$\left.\begin{aligned} D\sigma+E\varrho+F(s\varrho-r\sigma)+C &=0,\\ D\sigma+E\varrho+F(s\varrho-r\sigma)+Bs &=0,\\ D\sigma+E\varrho+F(s\varrho-r\sigma)+Ar &=0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

28. In order to represent a linear complex by equations of the utmost simplicity, let us take any plane XY, XZ, YZ perpendicular to the characteristic direction, and draw through its corresponding point O the axis OZ, OY, OX. The resulting equations will assume the following forms,

$$\left.\begin{aligned} F(s\varrho-r\sigma)+C &=0,\\ Bs \qquad +E\varrho &=0,\\ Ar \qquad +D\sigma &=0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23^*)$$

The planes corresponding to all points of a right line having the characteristic direction are parallel to each other; and conversely the locus of points corresponding to parallel planes is a right line of that direction. Hence we conclude that there is one fixed line, the points of which correspond to planes which are perpendicular to it. Consequently, on the supposition of rectangular coordinates, we may in only one way represent a linear complex by means of equations assuming the form of those above.

29. In order, for instance, to get the first of these equations, which by replacing $-\frac{C}{F}$ by $k$ may be written thus,

$$s\varrho-r\sigma=k,$$

it will be sufficient to direct OZ along the fixed line. As no supposition is made either with regard to the position of the origin on OZ, or to the direction of OX and OY within the plane XY which is perpendicular to OZ, this equation will remain absolutely the same if the system of coordinates be moved parallel to itself along OZ, or turned round it. In other terms,

*A linear complex of rays invariably remains the same if it be moved parallel to itself along a fixed right line or turned round it.*

The fixed right line may be called the *axis of rotation*, or merely the *axis* of the complex.

30. We may give different geometrical interpretations to the last three equations, involving each a characteristic property of a linear complex of rays.

Any two planes XZ, YZ intersecting each other along OZ being given, rays of space may be determined either by their projections on both planes, or by the points where they meet them. In the first case, if a third plane intersecting XZ, YZ along OX, OY at right angles be drawn, there are two planes LMN, L′M′N′, parallel to each other, passing through the two projections LN, M′N, and meeting OZ, OY, OX in N and N′, M and M′, L and L′. In the second supposition, denote the two points of intersection by U and V, and their projections by U′ and V′. Accordingly U′U, V′V, and U′V′ may be regarded as the projections of UV on the planes XZ, YZ, and on OZ. If in the

first case

$$\frac{\mathrm{LL}'.\,\mathrm{MM}'}{\mathrm{NN}'}=k,$$

in the second

$$\frac{\mathrm{UU}'.\,\mathrm{VV}'}{\mathrm{U}'\mathrm{V}'}=k,$$

all rays thus determined constitute the linear complex, represented by

$$s\varrho-r\sigma=k,$$

the axis of which is OZ.

If $k=0$, the linear complex is of a peculiar description, all its rays meet the same right line, the axis OZ.

31. The results of [29] may be derived in a direct way. Let $(x', y', z')$ be any point of space; according to the general equation (10) its corresponding plane with regard to the complex

$$s\varrho-r\sigma=k \quad \ldots \ldots \ldots \ldots \quad (24)$$

will be represented by

$$y'x-x'y=k(z-z'). \quad \ldots \ldots \ldots \ldots \quad (25)$$

In putting $x'=0$, $y'=0$, this equation shows that all planes corresponding to points of the axis of rotation OZ are perpendicular to this axis (in the case of oblique coordinates parallel to XY).

If the point fall within XY, we get by putting $z'=0$,

$$y'x-x'y=kz;$$

consequently the corresponding plane passes through O. In denoting the angle which it makes with the axis of rotation by $\lambda$, we obtain

$$\cos\lambda=\frac{k}{\sqrt{y'^2+x'^2+k^2}},$$

whence

$$y'^2+x'^2=k^2\tan^2\lambda. \quad \ldots \ldots \ldots \ldots \quad (26)$$

Hence we conclude,

Right lines parallel to and at an equal distance from the axis of the complex are met under the same angle by planes corresponding to their points.

32. The following results are immediately derived from (26).

The plane $p$ corresponding to any given point P passes through OP, O being the projection of P on OZ. Let the plane $p$ and the right line OM perpendicular to it in O turn round the axis OZ, through an angle $\frac{\pi}{2}$, and denote them after turning by $p'$ and OM$'$. The projection of OP on OM$'$ is a constant, and equal to $p$. So is the perpendicular drawn from P to $p'$.

Again, $k$ being given we may, by determining $\lambda$, construct the plane corresponding to a given point, and, conversely, by determining OP, construct the point corresponding to a given plane.

The following theorem is the geometrical interpretation of the equation (25).

Draw through a point P its corresponding plane $p$, and the plane XY perpendicular to the axis of the complex meeting that axis in O. Let R be an arbitrary point of $p$, and R′ its projection on XY. The double area of the triangle POR′ divided by R′R is a constant, and equal to $k$.

33. In order to generalize, we may start from the equation

$$\mathrm{A}r+\mathrm{B}s+\mathrm{C}+\mathrm{D}\sigma+\mathrm{E}\varrho+\mathrm{F}(s\varrho-r\sigma)=0 \quad . \quad (7),$$

and proceed in the following way. By replacing $x$, $y$, $z$ by $\xi$, $\eta$, $\zeta$, $\vartheta$ (see [16], note), and omitting the accents, we immediately derive from equation (10),

$$\left.\begin{aligned}\xi&=\ \ \mathrm{C}u-\mathrm{B}v-\mathrm{D}w,\\ \eta&=-\mathrm{C}t+\mathrm{A}v+\mathrm{E}w,\\ \zeta&=\ \ \mathrm{B}t-\mathrm{A}u-\mathrm{F}w,\\ \vartheta&=\ \ \mathrm{D}t-\mathrm{E}u+\mathrm{F}v,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$\xi$, $\eta$, $\zeta$, $\vartheta$ indicating any point, and $t$, $u$, $v$, $w$ its corresponding plane. From the first three of these equations results the equation

$$\mathrm{A}\xi+\mathrm{B}\eta+\mathrm{C}\zeta=-(\mathrm{AD}-\mathrm{BE}+\mathrm{CF})w,$$

which, multiplied member by member by the fourth equation,

$$\mathrm{D}t-\mathrm{E}u+\mathrm{F}v=\vartheta,$$

and divided by $\vartheta w$, furnishes the following relation,

$$(\mathrm{A}x+\mathrm{B}y+\mathrm{C}z)\left(\mathrm{D}\frac{t}{w}-\mathrm{E}\frac{u}{w}+\mathrm{F}\frac{v}{w}\right)=-(\mathrm{AD}-\mathrm{BE}+\mathrm{CF}). \quad . \quad . \quad . \quad (28)$$

In a similar way we obtain

$$\left.\begin{aligned}&\frac{(\mathrm{C}\vartheta+\mathrm{E}\xi+\mathrm{D}\eta)}{\zeta}\cdot\frac{\mathrm{B}t-\mathrm{A}u-\mathrm{F}w}{v},\\ =\ &\frac{\mathrm{B}\vartheta-\mathrm{D}\zeta+\mathrm{F}\xi}{\eta}\cdot\frac{-\mathrm{C}t+\mathrm{A}u+\mathrm{E}w}{u},\\ =\ &\frac{\mathrm{A}\vartheta-\mathrm{E}\zeta-\mathrm{F}\eta}{\xi}\cdot\frac{\mathrm{C}u-\mathrm{B}v-\mathrm{D}w}{t},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$=-(\mathrm{AD}-\mathrm{BE}+\mathrm{CF}).$$

34. In starting again from the equation (26),

$$s\varrho-r\sigma=k,$$

and in supposing that there is a right line determined by means of the coordinates of any two of its points $(x', y', z')$ and $(x'', y'', z'')$ according to [31], its conjugate line will be represented by the system of equations,

$$y'x-x'y=k(z-z'),$$
$$y''x-x''y=k(z-z''),$$

which, after eliminating successively $y$ and $x$, may be replaced by the following ones:

$$x''y'-x'y''=k[(x''-x')z-(x''z'-x'z'')],$$
$$x''y'-x'y''=k[(y''-y')z-(y''z'-y'z'')].$$

In denoting the coordinates of the two conjugate lines by

$$r_0,\ s_0,\ \varrho_0,\ \sigma_0, \text{ and } r^0,\ s^0,\ \varrho^0,\ \sigma^0,$$

the following relations are immediately obtained:

$$r_0=\frac{x''-x'}{z''-z'}, \qquad s_0=\frac{y''-y'}{z''-z'},$$

$$\varrho_0=\frac{x''z'-x'z''}{z''-z'}, \qquad \sigma_0=-\frac{y''z'-y'z''}{z''-z'},$$

$$s_0\varrho_0-r_0\sigma_0=-\frac{x''y'-x'y''}{z''-z'},$$

$$r^0=\ k\frac{x''-x'}{x''y'-x'y''}, \qquad s^0=\ k\cdot\frac{y''-y'}{x''y'-x'y''},$$

$$\varrho^0=-k\frac{x''z'-x'z''}{x''y'-x'y''}, \qquad \sigma^0=-k\frac{y''z'-y'z''}{x''y'-x'y''}.$$

Whence

$$\frac{r_0}{r^0}=\frac{s_0}{s^0}=\frac{\varrho_0}{\varrho^0}=\frac{\sigma_0}{\sigma^0}=\frac{(s_0\varrho_0-r_0\sigma_0)}{k}$$

and

$$(s_0\varrho_0-r_0\sigma_0)(s^0\varrho^0-r^0\sigma^0)=k^2.$$

Not any two conjugate right lines intersect each other; if congruent they belong to the complex.

35. A linear complex depends upon five constants, four of which fix in space the position of its axis. In the case of the equations (23), this axis falling within an axis of coordinates, there remains only one constant. The position of the axis of the complex and its remaining constant may be determined by means of the five independent constants of the general equation (7).

For that purpose we shall make use of the transformation of coordinates. If the axes of coordinates be changed, the coordinates of a ray change at the same time, and we get formulæ analogous to the formulæ in the case of ordinary coordinates, in order to express the coordinates of one system by means of the coordinates in the other.

36. Let

$$x=rz+\varrho,$$
$$y=sz+\sigma$$

be the equations of a ray referred to the system of coordinates $(x, y, z)$. If referred to another system $(x', y', z')$, its coordinates will be replaced by new ones $(r', s', \varrho', \sigma')$, but their equations retain the same shape,

$$x'=r'z'+\varrho',$$
$$y'=s'z'+\sigma'.$$

If the primitive system of coordinates be only displaced parallel to itself, the coordinates of the new origin being $(x^0, y^0, z^0)$, we obtain

$$x'=x-x^0, \quad y'=y-y^0, \quad z'=z-z^0;$$

and by substituting in the last equations,

$$x=r'z+(\varrho'+x^0-r'z),$$
$$y=s'z+(\sigma'+y^0-s'z);$$

whence, by comparison with the primitive equations,

$$\left.\begin{aligned} r&=r', \\ s&=s, \\ \varrho&=\varrho'+x^0-rz^0, \\ \sigma&=\sigma'+y^0-sz^0. \end{aligned}\right\} \quad \ldots\ldots \quad (30)$$

We have further

$$s\varrho-r\sigma=(s'\varrho'-r'\sigma')+x^0s-y^0r. \quad \ldots\ldots \quad (31)$$

If $x^0=0$, $y^0=0$, and accordingly the origin move along OZ, the expression $(s\varrho-r\sigma)$ remains unaltered [29].

37. If OY and OX turn round OZ, forming in the new position OY′, OX′ the angles $\alpha'$ and $\alpha$ with OX, we have

$$x=x'\cos\alpha+y'\cos\alpha'=rz+\varrho,$$
$$y=x'\sin\alpha+y'\sin\alpha'=sz+\sigma;$$

whence, on putting $(\alpha'-\alpha)=\vartheta$,

$$x'=\frac{r\sin\alpha'-s\cos\alpha'}{\sin\vartheta}z'+\frac{\varrho\sin\alpha'-\sigma\sin\alpha'}{\sin\vartheta},$$

$$y'=-\frac{r\sin\alpha-s\cos\alpha}{\sin\vartheta}\cdot z'-\frac{\varrho\sin\alpha-\sigma\sin\alpha'}{\sin\vartheta}.$$

We immediately derive from these equations of the ray in the new system $(x', y', z')$,

$$\left.\begin{aligned} r'\sin\vartheta&=r\sin\alpha'-s\cos\alpha', \\ \varrho'\sin\vartheta&=\varrho\sin\alpha'-\sigma\cos\alpha', \\ -s\sin\vartheta&=r\sin\alpha-s\cos\alpha, \\ -\sigma\sin\vartheta&=\varrho\sin\alpha-\sigma\cos\alpha, \end{aligned}\right\} \quad \ldots\ldots \quad (31^*)$$

whence

$$\left.\begin{aligned} r&=r'\cos\alpha+s'\cos\alpha', \\ \varrho&=\varrho'\cos\alpha+\sigma'\cos\alpha', \\ s&=r'\sin\alpha+s'\sin\alpha', \\ \sigma&=\varrho'\sin\alpha+\sigma'\sin\alpha', \end{aligned}\right\} \quad \ldots\ldots \quad (32)$$

and

$$(s\varrho-r\sigma)=(s'\varrho'-r'\sigma')\sin\vartheta. \quad \ldots\ldots \quad (33)$$

If especially $\vartheta=\frac{\pi}{2}$, the last four equations become

$$\left.\begin{aligned} r&=r'\cos\alpha-s'\sin\alpha,\\ \varrho&=\varrho'\cos\alpha-\sigma'\sin\alpha,\\ s&=r'\sin\alpha+s'\cos\alpha,\\ \sigma&=\varrho'\sin\alpha+\sigma'\cos\alpha, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (34)$$

and the expression

$$s\varrho-r\sigma$$

will not be altered by the transformation of coordinates [29].

38. Again, let OX and OZ turn round OY; let $\alpha'$ and $\alpha$ be the angles formed by these axes in their new position, OX$'$ and OZ$'$, with OZ, and $\alpha'-\alpha=\vartheta$. In the new system of coordinates the primitive equations of the ray become

$$(z'\sin\alpha+x'\sin\alpha')=(z'\cos\alpha+x'\cos\alpha')r+\varrho,$$
$$y'=(z'\cos\alpha+x'\cos\alpha')s+\sigma.$$

From the first of these equations we derive

$$x'(\sin\alpha'-r\cos\alpha')=-z'(\sin\alpha-r\cos\alpha')+\varrho,$$

whence

$$r'=-\frac{\sin\alpha-r\cos\alpha}{\sin\alpha'-r\cos\alpha'}, \quad \ldots\ldots\ldots \quad (35)$$

$$\varrho'=\frac{\varrho}{\sin\alpha'-r\cos\alpha'}. \quad \ldots\ldots\ldots \quad (36)$$

After replacing in the second equation of this number $x'$ by $(r'z'+\varrho')$, we obtain

$$y'=(\cos\alpha+r'\cos\alpha')sz'+(\sigma+s\varrho'\cos\alpha'),$$

whence

$$s'=(\cos\alpha+r'\cos\alpha')s,$$

$$\sigma'=\sigma+s\varrho'\cos\alpha';$$

and by eliminating $r'$ and $\varrho'$ by means of (35) and (36),

$$s'=\frac{s\sin\vartheta}{\sin\alpha'-r\cos\alpha'}, \quad \ldots\ldots\ldots \quad (37)$$

$$\sigma'=\frac{(\sigma\varrho-r\sigma)\cos\alpha'+\sigma\sin\alpha'}{\sin\alpha'-r\cos\alpha'}. \quad \ldots\ldots\ldots \quad (38)$$

From (35)-(38) we derive

$$s'\varrho'-r'\sigma'=\frac{(s\rho-r\sigma)\cos\alpha+\sigma\sin\alpha}{\sin\alpha'-r\cos\alpha'}; \quad \ldots\ldots\ldots \quad (39)$$

from (36) and (37),

$$\frac{\varrho'}{s'}=\frac{\varrho}{s}\sin\vartheta. \quad \ldots\ldots\ldots \quad (40)$$

On the supposition of rectangular axes of coordinates, the last equations become

$$\left.\begin{aligned} r' &= -\frac{\sin\alpha - r\cos\alpha}{\cos\alpha + r\sin\alpha}, \\ \varrho' &= \frac{\varrho}{\cos\alpha + r\sin\alpha}, \\ s' &= -\frac{s}{\cos\alpha + r\sin\alpha}, \\ \sigma' &= -\frac{(s\varrho - \varrho\sigma)\sin\alpha - \sigma\cos\alpha}{\cos\alpha + r\sin\alpha}, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (41)$$

$$s'\varrho' - r'\sigma' = \frac{(s\varrho - r\sigma)\cos\alpha - \sigma\sin\alpha}{\cos\alpha + r\sin\alpha}, \quad \ldots\ldots\ldots \quad (42)$$

$$\frac{\varrho'}{s'} = \frac{\varrho}{s}. \quad \ldots\ldots\ldots \quad (43)$$

In order to pass from the first system of coordinates to the second, $r$, $s$, $\varrho$, $\sigma$ and $r'$, $s'$, $\varrho'$, $\sigma'$ are to be replaced by one another, while the sign of $\alpha$ is to be changed. Thus we get the following formulæ:—

$$\left.\begin{aligned} r &= \frac{\sin\alpha + r'\cos\alpha}{\cos\alpha - r'\sin\alpha}, \\ \varrho &= \frac{\varrho'}{\cos\alpha - r'\sin\alpha}, \\ s &= -\frac{s'}{\cos\alpha - r'\sin\alpha}, \\ \sigma &= \frac{(s'\varrho' - r'\sigma')\sin\alpha + \sigma'\cos\alpha}{\cos\alpha - r'\sin\alpha}, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (44)$$

$$s\varrho - r\sigma = \frac{(s'\varrho' - r'\sigma')\cos\alpha - \sigma\sin\alpha}{\cos\alpha - r'\sin\alpha}. \quad \ldots\ldots\ldots \quad (45)$$

39. The general equation of the linear complex

$$Ar + Bs + C + D\sigma + E\varrho + F(s\varrho - r\sigma) = 0 \quad . \quad . \quad (7)$$

becomes, if the origin is moved to any point $(x^0, y^0, z^0)$ . . . (30),

$$(A - Fy^0 - Ez^0)r + (B + Fx^0 - Dz^0)s + (C + Ex^0 + Dz^0) + D\sigma' + E\varrho' + F(s\varrho - r\sigma) = 0.$$

If
$$\frac{x^0}{D} = \frac{y^0}{E} = \frac{z^0}{F},$$

the primitive equation is not altered. Consequently the complex remains the same if it be moved parallel to itself along a direction indicated by the last equations. We obtain in denoting by $\xi$, $\eta$, $\zeta$, the angles which this direction makes with OX, OY, OZ,

$$\frac{\cos\xi}{D} = \frac{\cos\eta}{E} = \frac{\cos\zeta}{F}. \quad \ldots\ldots\ldots \quad (46)$$

40. In order to get OZ congruent with a right line OM of the determined direction and passing through O, we may in the first instance turn the system of coordinates

round OZ in its primitive position through an angle $\alpha$ such that ZX in its new position contains OM. Accordingly we obtain

$$\cos\alpha=\frac{\cos\xi}{\sin\zeta},$$

whence

$$\tan^2\alpha=\frac{1-\cos^2\zeta-\cos^2\xi}{\cos^2\xi}=\frac{\cos^2\eta}{\cos^2\xi}=\frac{E^2}{D^2}.$$

By making use of the formulæ (34), the equation of the complex (7) becomes

$$\begin{gathered}(A\cos\alpha+B\sin\alpha)r'-(A\sin\alpha-B\cos\alpha)s'\\+(E\cos\alpha+D\sin\alpha)\varrho'-(E\sin\alpha-D\cos\alpha)\sigma'+C+F(s'\varrho'-r'\sigma')=0,\end{gathered}$$

and may be written thus,

$$A'r+B's+C'+D'\sigma+F'(s\varrho-r\sigma)=0, \quad \ldots \ldots \quad (47)$$

in omitting the accents of the new coordinates and in putting

$$\left.\begin{aligned}&E\cos\alpha+D\sin\alpha=0,\\&A'=(AD-BE)\frac{\cos\alpha}{D},\quad B'=(AD+BE)\frac{\cos\alpha}{D},\\&D'=(D^2+E^2)\frac{\cos\alpha}{D},\quad C'=C,\quad F'=F.\end{aligned}\right\} \quad \ldots \ldots \quad (48)$$

41. In order to give within ZX to OZ the required direction along OM, the formulæ (44) are to be used after having replaced $\alpha$ by $\zeta$. Accordingly the equation (47) is transposed into the following one,

$$\begin{gathered}A'(\sin\zeta+r'\cos\zeta)-B's'+C'(\cos\zeta-r'\sin\zeta)\\+D'\big((s'\varrho'-r'\sigma')\sin\zeta+\sigma'\cos\zeta\big)+F'\big((s'\varrho'-r'\sigma')\cos\zeta-\sigma'\sin\zeta\big)=0,\end{gathered}$$

and may be written thus,

$$A''r+B''s+C''+F''(s\varrho-r\sigma)=0, \quad \ldots \ldots \quad (48^*)$$

on omitting the accents of the coordinates and putting

$$\left.\begin{aligned}&D'\cos\zeta=F'\sin\zeta,\\&A''=(A'F'-C'D')\frac{\cos\zeta}{F'},\\&B''=-B',\\&C''=(A'F'+A'D')\frac{\cos\zeta}{F'},\\&F''=(D'^2+F'^2)\frac{\cos\zeta}{F'}.\end{aligned}\right\} \quad \ldots \ldots \quad (49)$$

42. Finally, the origin may be moved within XY to a point the coordinates of which are $x^0$ and $y^0$. Accordingly the equation of the complex, on replacing $\varrho$ and $\sigma$ by $\varrho+x^0$ and $\sigma+y^0$, becomes

$$(A''-F''y^0)r+(B''+F''x^0)s+C''+F''(s\varrho-r\sigma)=0,$$

and by putting

$$y^0=\frac{A''}{F''},\quad x^0=-\frac{B''}{F''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

is reduced to

$$(s\varrho-r\sigma)=-\frac{C''}{F''}=k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

43. By successive substitution we obtain

$$\begin{aligned} k&=-\frac{C''}{F''}\\ &=-\frac{C'F'+A'D'}{D'^2+F'^2}\\ &=-\frac{CF+(AD-BE)(D^2+E^2)\frac{\cos^2\alpha}{D^2}}{(D^2+E^2)^2\frac{\cos^2\alpha}{D^2}+F^2}; \end{aligned}$$

and finally, on observing that

$$\cos\alpha^2=\frac{D^2}{D^2+E^2},$$

the symmetrical expression

$$k=-\frac{AD-BE+CF}{D^2+E^2+F^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

In order to replace OZ and OX by each other, we may make use of the formulæ (41) and (42) on putting $\alpha=\frac{1}{2}\pi$. By means of the last of these formulæ the equation of the linear complex (51) is immediately transformed into the following one,

$$\sigma=kr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

the constant $k$ being the same as before.

Again, on interchanging OY and OX, we get

$$\varrho=ks. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

44. If $k$ become equal to zero the complex is of a peculiar description, all its rays meet a fixed line. If the complex be represented by the general equation

$$Ar+Bs+C+D\sigma+E\varrho+F(s\varrho-r\sigma)=0, \quad . \quad . \quad . \quad (7)$$

this peculiar case is indicated by the following condition,

$$AD-BE+CF=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

45. By eliminating from the general equation of the complex $\sigma$, $\varrho$ and $(s\varrho-r\sigma)$ by means of the equations

$$\begin{aligned} x&=rz+\varrho,\\ y&=sz+\sigma,\\ sx-ry&=s\varrho-r\sigma. \end{aligned}$$

we get

$$(A+Fy-Ez)r+(B+Fx-Dz)s+(C+Dy+Ex)=0.$$

If there exist a point $(x, y, z)$ where all rays of the complex meet, this point will be determined by means of the following three equations,

$$\left.\begin{aligned} A-Fy-Ez&=0,\\ B+Fx-Dz&=0,\\ C+Ex+Dy&=0.\end{aligned}\right\} \quad \ldots \ldots \quad (56)$$

These three equations can subsist simultaneously only in the case where (55) is satisfied.

If this condition be satisfied, the locus of points, where all rays of the complex meet, is a right line, the projections of which are represented by the last equations (56).

46. Such rays as belong to both linear complexes,

$$\left.\begin{aligned} \Omega &\equiv Ar+Bs+C+D\sigma+E\varrho+F(s\varrho-r\sigma)=0,\\ \Omega' &\equiv A'r+B's+C'+D'\sigma+E'\varrho+F'(s\varrho-r\sigma)=0,\end{aligned}\right\} \quad \ldots \ldots \quad (57)$$

constitute a *linear congruency of rays represented by the system of the two equations.* In order to determine the congruency each of the two complexes,

$$\Omega=0, \qquad \Omega'=0$$

may be replaced by any other represented by

$$\Omega+\mu\Omega'=0, \quad \ldots \ldots \quad (58)$$

where arbitrary values are given to the coefficient $\mu$.

In each of the two complexes by means of which the congruency is determined, there is a plane corresponding to each point of space which contains all rays starting from that point. Both planes corresponding to the same point intersect each other along a single ray, belonging to both complexes, *i. e.* to the congruency. *With regard to the congruency one ray corresponds to a given point of space.* The planes corresponding to the same point, in all complexes, represented by (58) meet along a fixed line, the corresponding ray of the congruency.

Conversely, there is in each of the complexes (58) a point corresponding to a given plane in which all rays confined within the plane meet. By means of two such complexes we get, within the given plane, two points; the right line joining the two points is the only ray of the plane common to both complexes, and therefore belonging to the congruency. We call it *the ray of the congruency corresponding to the given plane.*

To each point, as well as to each plane, corresponds only one ray. There are not any two rays of the congruency intersecting one onother, or, in other terms, confined within the same plane.

47. Suppose that AB is any given right line, and A′B′, A″B″ its two conjugate with regard to the complexes $\Omega$, $\Omega'$. Let C be any point of AB. Each ray starting from C, if confined within the plane A′B′C belongs to $\Omega$, if confined within A″B″C to $\Omega'$. Therefore the intersection of the two planes A′B′C, A″B″C, *i. e.* the right line starting from C and meeting both conjugate, is the ray of the congruency which corresponds to the point C. If C move along AB, all rays of the congruency obtained in that way are the

rays of one generation of a *hyperboloid*, while the given right line AB and its two conjugate A'B', A''B'' are rays of its other generation. In replacing $\Omega$ and $\Omega'$ by other complexes arbitrarily taken among the complexes (58), the conjugate will be replaced by others, all intersected by the rays of the congruency starting from AB. Hence

*The right lines conjugate to a given one, with regard to all complexes intersecting one another along a linear congruency, belong to one generation of a hyperboloid, while the right lines of its second generation are rays of the congruency meeting the given line.*

48. If a point move along a given right line of space, according to the last number, its corresponding ray generally describes a hyperboloid. We may say that the same hyperboloid is described by the ray which corresponds to a plane passing through the given right line and turning round it. If the ray be the same in both cases, the point where it meets the given line AB is a point of the surface, and the plane confining both AB and the ray, the tangent plane in that point.

49. The hyperboloid generated by a ray of a linear congruency, the corresponding point of which moves along AB, varies if this line turn round one of its points C. All the new hyperboloids contain the ray which corresponds to C, but there is no other ray common to any two of them. If AB describe a plane, by turning round C through an angle $\pi$, there will be one ray of a hyperboloid passing through any point of space. A linear congruency therefore may be generated by a variable hyperboloid turning round one of its rays.

In an analogous way, a linear complex may be generated by a revolving variable congruency.

50. While in each of the two complexes $\Omega$ and $\Omega'$ there is a fixed line—the axis of the complex around which its rays are symmetrically distributed—there is in a linear congruency a characteristic section parallel to both axes of the complexes, and a characteristic direction perpendicular to it.

The characteristic section, if conducted through the origin O, may be represented by the equation

$$ax+by+cz=0.$$

The two right lines starting from O and parallel to the two axes of the complexes are represented by the double equations,

$$\frac{x}{D}=\frac{y}{E}=\frac{z}{F},$$

$$\frac{x}{D'}=\frac{y}{E'}=\frac{z}{F'}.$$

These lines being confined within the section, we get in order to determine the constants of its equation,

$$aD+bE+cF=0,$$

$$aD'+bE'+cF'=0,$$

whence

$$(D'E-E'D)b+(D'F-F'D)c=0,$$

$$(D'E-E'D)a-(E'F-F'E)c=0.$$

Accordingly the equation of the section becomes

$$(E'F-F'E)x-(D'F-F'D)y+(D'E-E'D)z=0, \quad \ldots \ldots \quad (59)$$

and the double equation of the right line perpendicular to it,

$$\frac{x}{E'F-F'E}=\frac{-y}{D'F-F'D}=\frac{z}{D'E-E'D}. \quad \ldots \ldots \quad (60)$$

51. By giving to OZ the characteristic direction, the two complexes (57) will be represented by linear equations of the form

$$\left.\begin{aligned}\Omega &\equiv Ar+Bs+C+D\sigma+E\varrho=0,\\ \Omega' &\equiv A'r+B's+C'+D'\sigma+E'\varrho=0,\end{aligned}\right\} \quad \ldots \ldots \quad (61)$$

the origin and the direction of OX and OY, perpendicular to OZ, remaining arbitrary.

Again, OZ may be moved parallel to itself, and accordingly $\varrho$ and $\sigma$ replaced by $(\varrho+x^0)$ and $(\sigma+y^0)$, $x^0$ and $y^0$ being the coordinates of the new origin. If especially

$$C+Dy^0+Ex^0=0,$$
$$C'+D'y^0+E'x^0=0,$$

whence

$$x^0=-\frac{C'D-D'C}{D'E-E'D},$$
$$y^0=\frac{C'E-E'C}{D'E-E'D};$$

by the mere disappearance of C and C′ the equations of the two complexes become

$$\left.\begin{aligned}\Omega &\equiv Ar+Bs+D\sigma+E\varrho=0,\\ \Omega' &\equiv A'r+B's+D'\sigma+E'\varrho=0.\end{aligned}\right\} \quad \ldots \ldots \quad (62)$$

OZ in its new position is a completely determined right line, which may be called the *axis of the congruency*. It is easily seen that it intersects at right angles the two axes of rotation of the complexes $\Omega$ and $\Omega'$, and consequently the axes of all complexes represented by (58).

52. The planes corresponding in the two complexes (62) to a given point $(x', y', z')$ are represented by

$$\left.\begin{aligned}(A-Ez')x+(B-Dz')y+(Ex'+Dy')z&=Ax'+By',\\ (A'-E'z')x+(B'-D'z')y+(E'x'+D'y')z&=A'x'+B'y'.\end{aligned}\right\} \quad \ldots \ldots \quad (63)$$

In order to express that both corresponding planes are the same, we obtain the following relations,

$$\left.\begin{aligned}(A-Ez'):(B-Dz'):(Ex'+Fy'):(Ax'+By')&=\\ (A'-E'z'):(B'-D'z'):(E'x'+F'y'):(A'x'+B'y').&\end{aligned}\right\} \quad \ldots \ldots \quad (64)$$

Since both planes pass through the given point, any two equations, hence derived, are sufficient in order to determine the locus of points having, in both complexes, the same corresponding plane. From any two of the following six equations where the accents are omitted, the remaining four may be derived:

$$(D'E-E'D)z^2-[(B'E-E'B)-(A'D-D'A)]z-(A'B-B'A)=0, \quad . \quad . \quad (65)$$
$$(B'D-D'B)y^2+[(B'E-E'B)+(A'D-D'A)]xy+(A'E-E'A)x^2=0, \quad . \quad . \quad (66)$$
$$(A'D-D'A)y+(A'E-E'A)x+(D'E-E'D)yz=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$
$$(B'D-D'B)y-(B'E-E'B)x-(D'E-E'D)xz=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$
$$(A'B-B'A)y+(A'E-E'A)xz-(B'E-E'B)yz=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$
$$(A'B-B'A)x-(A'D-D'A)xz+(B'D-D'B)xz=0^{*}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

53. According to the first two equations (65), (66), the locus in question is a system of two right lines both intersecting OZ. These lines are confined within two planes parallel to XY and determined by (65); their direction within these planes is given by (66). We shall call them the "*directrices*," and the characteristic section parallel to both and equidistant from them, *the central plane of the linear congruency.* Both "directrices" intersect at right angles the axis of the congruency, as the axes of all complexes do.

54. We may distinguish two general classes of linear congruencies; either both directrices are *real* or both *imaginary*. In a particular case the two directrices are congruent. Finally, one of the two directrices may pass at an infinite distance.

55. If the directrices are real, and the plane XY be conducted through one of them, the following condition,

$$A'B-B'A=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

is derived from (65). In order to determine within XY the direction of that directrix, we get from (67), by putting $z=0$,

$$(A'D-D'A)y+(A'E-E'A)x=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

There is among the infinite number of complexes containing the congruency, which are represented by

$$\Omega+\mu\Omega'=0,$$

one of a particular description. It is obtained if, starting from (62), we put

$$\mu=-\frac{A}{A'}=-\frac{B}{B'};$$

whence

$$(A'D-D'A)\sigma+(A'E-E'A)\varrho=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

All rays of that complex, and therefore all rays of the congruency, meet within XY a fixed right line, represented by (72), on replacing $\varrho$ and $\sigma$ by $x$ and $y$. This line therefore is the axis of that complex, and one of the two directrices of the congruency. In the same way it may be proved that likewise all rays of the congruency meet the other directrix. Hence

*All rays of a congruency meet its two directrices.*

* We may observe that any equation which, like those above, is homogeneous with regard to (A'B—B'A), A'C—C'A) ... will not be altered if the complexes $\Omega$ and $\Omega'$ are replaced by any of the complexes $(\Omega+\mu\Omega')$.

Accordingly, both directrices being real and known, we may immediately draw through any given point the only corresponding ray of the congruency.

56. In that peculiar class of congruencies indicated by the condition

$$D'E-E'D=0, \quad \ldots \ldots \ldots \quad (74)$$

one of the two directrices passes at an infinite distance. By putting simultaneously

$$A'B-B'A=0,$$

we get, in order to represent the only remaining directrix, now confined within XY, the same equation as before (72). But among the complexes,

$$\Omega+\mu\Omega'=0,$$

there is, besides the complex (73), the axis of which is the directrix, another complex, represented by

$$D\Omega'-D'\Omega\equiv(A'D-D'A)r+(B'D-D'B)s=0,$$

the rays of which are parallel to a given plane. Its equation may be transformed into

$$Ar+Bs=0; \quad \ldots \ldots \ldots \quad (75)$$

accordingly the equation of the plane becomes

$$Ax+By=0.$$

Hence in this peculiar case

*All rays of the linear congruency meet the only directrix, and are parallel to a given plane.*

57. From the last considerations we conclude that among the complexes intersecting each other along a linear congruency, and represented by

$$\Omega+\mu\Omega'=0, \quad \ldots \ldots \ldots \quad (76)$$

there are in the general case two, of a peculiar description, all the rays of which meet their axes. These axes, the directrices of the congruency, are two conjugate right lines with regard to each of the complexes (76).

Generally there is only one ray of the congruency passing through a given point, as there is only one ray confined within a given plane. But each of the two directrices may be considered as the locus of points, from which start an infinite number of rays, constituting a plane which passes through the other directrix. It may be likewise regarded as enveloped by planes, confining each an infinite number of rays, which converge towards a point of the other directrix.

58. We may represent *any two* complexes $\Omega$, $\Omega'$ in any position whatever by equations depending only upon the position of their axes and their constants. Let $\Delta$ be the shortest distance of the two axes from each other, and $\vartheta$ the angle between their directions.

Suppose that OZ intersects at right angles the axes of both complexes. Let OX be the axis of the first complex $\Omega$, $k$ its constant, OX perpendicular to XZ. The equation of the complex will be

$$\sigma=kr.$$

If the axis OY be turned round O till, in its new position OY′, the angle Y′OX becoming $\vartheta$, the plane ZOY′ passes through the axis of the second complex, the last equation, by putting

$$\sigma=\sigma'\sin\vartheta,$$
$$r=r'+s'\cos\vartheta,$$

assumes the following form,

$$\sigma'\sin\vartheta=kr'+ks'\cos\vartheta.$$

The axis of the second complex $\Omega'$ meets OZ in a point O′, O′O being $\Delta$. O′ may be regarded as the origin of new coordinates, OY and OZ being replaced by O′Y″ congruent with the axis of $\Omega'$, and by O′X″ perpendicular to ZY″; then the second complex $\Omega'$ will be represented by the equation

$$\varrho''=k's'',$$

$\varrho''$ and $s''$ being the new ray-coordinates and $k'$ the constant of the complex. In order to make O′X″ parallel to OX′, it is to be turned round O′ till, in its new position O′X‴, the angle Y‴O′X″ becomes $\vartheta$. Accordingly, by putting

$$\varrho''=\varrho'''\sin\vartheta,$$
$$s''=r'''\cos\vartheta+s''',$$

the equation of the complex is transformed into the following,

$$\varrho'''\sin\vartheta=k'r'''\cos\vartheta+k's'''.$$

Finally, by displacing the origin O′ into O, $\varrho'''$ becomes $\varrho^{\text{IV}}+\Delta r'''$, whence

$$\varrho'''\sin\vartheta=(k'\cos\vartheta+\Delta\sin\vartheta)r'''+k's'.$$

On omitting the accents, both complexes $\Omega$ and $\Omega'$, referred to the same axes of coordinates OZ, OY′, OX, the two last of which include an angle $\vartheta$, are represented by the following equations,

$$\left.\begin{aligned}\sigma\sin\vartheta&=kr+k\cos\vartheta.s,\\ \varrho\sin\vartheta&=(k'\cos\vartheta+\Delta\sin\vartheta)r+k's.\end{aligned}\right\}\quad\ldots\ldots\quad(77)$$

59. In order to determine the directrices of the congruency represented by the system of the last equations (77), the equations (65) and (66) may be transformed by putting

$$\begin{array}{llll} A=k, & B=k\cos\vartheta, & D=-\sin\vartheta, & E=0,\\ A'=k'\cos\vartheta+\Delta\sin\vartheta, & B'=k', & D'=0, & E'=-\sin\vartheta \end{array}$$

into those following,

$$0=(z\sin\vartheta)^2-[(k+k')\cos\vartheta+\Delta\sin\vartheta]z\sin\vartheta+(kk'\sin^2\vartheta-\Delta k\sin\vartheta\cos\vartheta),\quad .\quad(78)$$

$$0=\left(\frac{y}{x}\right)^2-\frac{(k'-k)\cos\vartheta-\Delta\sin\vartheta}{k'}\cdot\frac{y}{x}-\frac{k}{k'}.\quad\ldots\ldots\quad(79)$$

On denoting ıe roots of these equations by $z'\sin\vartheta$, $z''\sin\vartheta$, and $\left(\frac{y}{x}\right)'$, $\left(\frac{y}{x}\right)''$, we obtain

$$z'+z'' = \frac{(k-k')\cos\vartheta+\Delta\sin\vartheta}{\sin\vartheta},$$

$$(z'-z'')^2 = \frac{4kk'+[(k-k')\cos\vartheta-\Delta\sin\vartheta]^2}{\sin^2\vartheta},$$

$$\left(\frac{y}{x}\right)'+\left(\frac{y}{x}\right)'' = \frac{(k+k')\cos\vartheta-\Delta\sin\vartheta}{k'},$$

$$\left(\left(\frac{y}{x}\right)'-\left(\frac{y}{x}\right)''\right)^2 = \frac{4kk'+[(k-k')\cos\vartheta-\Delta\sin\vartheta]^2}{k'^2}.$$

The roots of both equations are simultaneously either real, or imaginary, or congruent. In the last case we have

$$(k-k')\cos\vartheta-\Delta\sin\vartheta=2\sqrt{-kk'},$$

whence

$$\left(\frac{y}{x}\right)'=\left(\frac{y}{x}\right)''=\sqrt{-\frac{k}{k'}}.$$

The central plane of the congruency is represented by

$$z=\frac{(k-k')\cos\vartheta-\Delta\sin\vartheta}{2\sin\vartheta}. \qquad (80)$$

In two peculiar cases this equation becomes

$$z=\tfrac{1}{2}\Delta,$$

either if

$$\vartheta=\tfrac{1}{2}\pi,$$

or, whatever may be $\vartheta$, if

$$k=k'.$$

Hence the axes of any two complexes selected among those intersecting each other along a given congruency are at equal distances from its central plane if their directions are perpendicular to each other, or if the constants of both complexes are the same.

60. Without entering into a more detailed discussion of the last results we may finally treat the inverse problem: a congruency being given by means of its two directrices, to determine the complexes passing through it. On the supposition of rectangular coordinates, the two directrices may be represented by the following systems of equations,

$$y-ax=0, \qquad z=\theta,$$
$$y+ax=0, \qquad z=-\theta.$$

These directrices are the axes of two complexes of a peculiar description, ranging among the infinite number of complexes which intersect each other along the congruency. The two complexes, if moved parallel to themselves till their axes fall within XY, are represented by the equations

$$\sigma-a\varrho=0,$$
$$\sigma+a\varrho=0,$$

whence, in order to represent them in their primitive position, the following equations are derived,

$$\sigma - a\varrho + \theta s - \theta a r = 0,$$

$$\sigma + a\varrho - \theta s - \theta a r = 0.$$

By adding the two equations, after having multiplied the second by an undetermined coefficient $\mu$, the following equation results,

$$(1+\mu)\sigma - (1-\mu)a\varrho + (1-\mu)\theta s - (1+\mu)\theta a r = 0,$$

which, on putting

$$\frac{1-\mu}{1+\mu} = \lambda,$$

becomes

$$\sigma - \lambda a\varrho + \lambda\theta s - \theta a r = 0. \quad \ldots \quad (81)$$

By varying $\lambda$ all complexes intersecting each other along the congruency are represented by this equation. Their axes are parallel to XY and meet OZ. According to (19) and (52) we may immediately derive the direction of the axes and their constants. The following way of proceeding leads us to the same results, giving besides the position in space of their axes.

By turning OX and OY round OZ through an angle $\omega$, by means of the formula (34), in which $\alpha$ is to be replaced by $\omega$, the last equation is transformed into the following one,

$$(\cos\omega + \lambda a \sin\omega)\sigma' + (\sin\omega - \lambda a \cos\omega)\varrho' + (\lambda\cos\omega + a\sin\omega)\theta s' + (\lambda\sin\omega - a\cos\omega)\theta r' = 0,$$

whence, by putting

$$\tan\omega = \lambda a, \quad \ldots \quad (82)$$

we obtain

$$(1+\tan^2\omega)\sigma' + (\lambda\tan\omega - a)\theta r' + (\lambda + a\tan\omega)\theta s' = 0.$$

Finally, by displacing the system of coordinates parallel to itself in such a way that the origin moves along OZ through $z^0$, we get

$$(1+\tan^2\omega)\sigma' + (\lambda\tan\omega - a)\theta r' + (\lambda + a\tan\omega)\theta s' - (1+\tan^2\omega)z^0 s' = 0,$$

whence, by putting

$$z^0 = \frac{\lambda + a\tan\omega}{1+\tan^2\omega} \cdot \theta, \quad \ldots \quad (83)$$

there results

$$\sigma' = -\frac{\lambda\tan\omega - a}{1+\tan^2\omega} \cdot \theta r' = k r'. \quad \ldots \quad (84)$$

The values of $\tan\omega$, $z^0$, and $k$ remain real if both directrices become imaginary. In this case, XY always remaining the central plane of the congruency and OZ its axis, $a$, $\theta$, and $\mu$ are to be replaced by $a\sqrt{-1}$, $\theta\sqrt{-1}$, $\mu\sqrt{-1}$. If $a$ be real, we may put

$$a = \tan\alpha,$$

$2\alpha$ being the angle between the directions of the two directrices, bisected by XZ. Accordingly we get

$$\lambda = \frac{\tan\omega}{\tan\alpha}, \quad \ldots \quad (85)$$

$$\left.\begin{aligned} z^0 &= \theta\frac{1+\tan^2\alpha}{\tan\alpha}\cdot\frac{\tan\omega}{1+\tan^2\omega} \\ &= \theta\frac{\sin\omega\cos\omega}{\sin\alpha\cos\alpha} \\ &= \theta\frac{\sin 2\omega}{\sin 2\alpha}, \end{aligned}\right\} \quad \ldots \quad (86)$$

$$\left.\begin{aligned} k &= \theta\frac{\tan^2\alpha - \tan^2\omega}{\tan\alpha(1+\tan^2\omega)} \\ &= \theta\frac{\sin^2\alpha\cos^2\omega - \sin^2\omega\cos^2\alpha}{\sin\alpha\cos\alpha} \\ &= \theta\frac{\sin(\alpha+\omega)\sin(\alpha-\omega)}{\sin\alpha\cos\alpha}. \end{aligned}\right\} \quad \ldots \quad (87)$$

The expression of $z^0$ shows that the axis within the central plane is directed along one of the two right lines bisecting, within this plane, the angle between the directions of the two directrices. These two right lines, having a peculiar relation to the congruency, may be called its *second* and *third* axis. The three axes, perpendicular to each other, meet in the *centre* of the congruency.

In order to express the angle $\omega$ by means of $z^0$, we get the following equation,

$$\sin 2\omega = \frac{z^0}{\theta}\sin 2\alpha,$$

indicating two directions perpendicular to each other, and corresponding to any value of $z^0$.

61. By replacing in the expression

$$z^0 = \frac{\theta}{\sin\alpha\cos\alpha}\cdot\frac{\tan\omega}{1+\tan^2\omega}$$

$\tan\omega$ by $\frac{y}{x}$, we obtain on omitting the accent of $z^0$,

$$z(y^2+x^2) = \frac{\theta}{\sin\alpha\cos\alpha}\cdot xy. \quad \ldots \quad (88)$$

The axes of all complexes constituting the congruency are confined within the surface represented by that equation. But this equation remaining unaltered if the axes OX and OY are replaced by one another, it is evident that the same surface contained the axes of two different series of complexes; one of the two series constituting the given congruency, while the other constitutes a strange one, obtained by turning the given congruency round its axis through a right angle.

62. In representing any three linear complexes by

$$\left.\begin{aligned}\Omega &\equiv \mathrm{A}r+\mathrm{B}s+\mathrm{C}+\mathrm{D}\sigma+\mathrm{E}\varrho+\mathrm{F}(s\varrho-r\sigma)=0,\\ \Omega' &\equiv \mathrm{A}'r+\mathrm{B}'s+\mathrm{C}'+\mathrm{D}'\sigma+\mathrm{E}'\varrho+\mathrm{F}'(s\varrho-r\sigma)=0,\\ \Omega'' &\equiv \mathrm{A}''r+\mathrm{B}s''+\mathrm{C}''+\mathrm{D}''\sigma+\mathrm{E}''\varrho+\mathrm{F}''(s\varrho-r\sigma)=0,\end{aligned}\right\} \quad \ldots\ldots \quad (89)$$

the system of these three equations represents a *linear configuration of rays*. The complexes may be replaced by any three selected among those represented by

$$\Omega+\mu\Omega'+\nu\Omega''=0$$

on giving to $\mu$ and $\nu$ any values whatever. By combining the three complexes $\Omega$, $\Omega'$, $\Omega''$ we get three congruencies, and accordingly three couples of directrices. Each ray of the configuration, belonging simultaneously to the three congruencies, meets both directrices of each couple. Hence in the general case the configuration is a *hyperboloid*; *its rays constitute one of its generations, while the directrices of all congruencies passing through it are right lines of its other generation.* Any three directrices are sufficient in order to determine the hyperboloid.

63. Let P and P′, Q and Q′, R and R′ be the three couples of directrices, each couple determining a central plane. The three central planes $\Pi$, K, P meet in one point C, which shall be called the *centre* of the configuration. The segment of any ray of a congruency bounded by both directrices being bisected by the central plane, the three right lines drawn through the centre C of the configuration to the three couples of directrices are bisected in the centre; they may be called *diameters* of the configuration.

Let, for instance, $\pi$ and $\pi'$ be the extremities of that diameter, $\pi\mathrm{C}\pi'$, which meets both directrices P and P′. The ray of the congruency $(\Omega, \Omega')$ passing through $\pi$ is parallel to P′, the ray passing through $\pi'$ parallel to P. Both planes $p$ and $p'$, drawn through P and P′ parallel to the central plane $\Pi$, each confining two right lines (one directrix and the ray parallel to the other) which belong to the two generations of the hyperboloid, touch that configuration, and the point where both right lines in each plane meet is the point of contact.

Draw through the six directrices P and P′, Q and Q′, R and R′ six planes $p$ and $p'$, $q$ and $q'$, $r$ and $r'$ parallel to the central planes $\Pi$, K, P. The six planes thus obtained constitute a paralellopiped circumscribed to the configuration, the three diameters of which join each the points of contact within two opposite planes. The axes of the three corresponding congruencies $(\Omega, \Omega')$, $(\Omega, \Omega')$, $(\Omega', \Omega'')$ are equal to the distance of the three couples of opposite planes; their centres are easily found.

64. The hyperboloid thus obtained is not changed if the complexes $\Omega$, $\Omega'$, $\Omega''$ be replaced by any three others taken among the complexes

$$\Omega+\mu\Omega'+\nu\Omega''=0,$$

but the three congruencies vary, and their directrices and the three diameters of the hyperboloid. The directrices may be either real or imaginary; accordingly the three

diameters either intersect the hyperboloid or do not meet it. In the intermediate case, where both congruencies are congruent, the corresponding diameter falls within the asymptotic cone of the surface.

65. Conversely, starting from the hyperboloid and any three of its diameters, we may revert to the three corresponding congruencies and the series of complexes by means of which these congruencies are determined. If especially the three diameters are the axes of the hyperboloid, the axes of the three congruencies meet in the same point, the centre of the surface, and are directed along its axes.

There is a double way of reverting from a given hyperboloid to the congruencies, and further on to the complexes. The right lines constituting each of its two generations may be considered as its rays, while the right lines of its other generation will be found to be the directrices of the congruencies passing through the surface.

66. It might be desirable to support in the analytical way the geometrical results explained in the last numbers. For that purpose we may select in order to determine the configuration, three complexes of that peculiar description where all rays meet the axis. Accordingly the axes of the three complexes $\Omega$, $\Omega'$, $\Omega''$ are three of the six directrices, P, Q, R for instance, confined within the planes $p$, $q$, $r$. In assuming these planes as planes of coordinates XY, XZ, YZ, the three complexes, constituting the configuration, are represented by equations of the following form,

$$\left.\begin{aligned} \Omega &\equiv C \quad +D\sigma +E\varrho=0, \\ \Omega' &\equiv B's +D'\sigma +F'(s\varrho-r\sigma)=0, \\ \Omega'' &\equiv A''r+E''\varrho+F''(s\varrho-r\sigma)=0. \end{aligned}\right\} \quad \ldots \quad (90)$$

In order to represent by means of a single equation between $x$, $y$, $z$ a configuration determined by means of three equations between ray-coordinates, these coordinates are to be eliminated by means of the following two equations,

$$x=rz+\varrho,$$
$$y=sz+\sigma,$$

to which the third derived one,

$$sx-ry=s\varrho-r\sigma,$$

may be added. In our case we may at first eliminate $s\varrho-r\sigma$, whence

$$(B' +Fx' )s-F'yr +D'\sigma =0,$$
$$(A''-F''y)r+F''xs+E''\varrho=0,$$

and after that $\varrho$ and $\sigma$,

$$Ezr+Dzs=C+Dy+Ex,$$
$$(B' +F'x -D'z )s-F'yr +D'y =0,$$
$$(A'' -F''y-E''z)r+F''xs+E''x=0.$$

Finally, by putting the values of $r$ and $s$ taken from the last two equations into the first one, we obtain

$$\begin{aligned}&\{(B'+F'x-D'z)E''-F''D'y\}Exz\\ &+\{(A''-F''y-E''z)D'-E''F'x\}Dyz\\ &+\{(A''-F''y-E''z)(B'+F'x-D'z)+F'F''xy\}(C+Dy+Ex)=0,\end{aligned}$$

which, by the disappearance of terms of the third order, becomes

$$\left.\begin{aligned}&A''B'C+A''(B'E+CF')x+B'(A''D-CF'')y-C(A''D'+E''B')z\\ &\quad+A''F'Ex^2-B'F''Dy^2+CE''D'z^2\\ &+(A''F'D-B'F''E)xy-(A''D'E+CE''F)xy\\ &\quad+(CF''D'-B'E''D)yz=0.\end{aligned}\right\} \quad . \quad . \quad (91)$$

After dividing by $A''B'C$ and replacing

$$-\frac{E}{C}, -\frac{D}{C}, \frac{D'}{B'}, -\frac{F'}{B'}, \frac{E''}{A''}, \frac{F''}{A''}$$

by $\xi$, $\eta$, $\zeta'$, $\xi'$, $\zeta''$, $\eta''$, the last equation assumes the following symmetrical form,

$$\left.\begin{aligned}&1-(\xi+\xi')x-(\eta+\eta'')y-(\zeta'+\zeta'')z\\ &\quad+\xi\xi'x^2+\eta\eta''y^2+\zeta'\zeta''z^2\\ &+(\xi'\eta+\xi\eta'')xy+(\xi'\zeta''+\xi\zeta')xz+(\eta\zeta'+\eta''\zeta')yz=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (92)$$

In order to represent the configuration this equation replaces the three equations (90), which may be written thus,

$$\left.\begin{aligned}&\eta\sigma+\xi\varrho-1=0,\\ &\zeta'\sigma-\xi'(s\varrho-r\sigma)-1=0,\\ &\zeta''\varrho-\eta''(s\varrho-r\sigma)+1=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

It shows that the configuration is a hyperboloid touching the three planes XY, XZ, YZ. The rays within these planes are represented by

$$\left.\begin{aligned}z&=0, & \xi x+\eta y&=1,\\ y&=0, & \xi'x+\zeta'z&=1,\\ x&=0, & \eta''y+\zeta''z&=1,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

the directrices within them by

$$\left.\begin{aligned}z&=0, & \xi'x+\eta''y&=1,\\ y&=0, & \xi x+\zeta''z&=1,\\ x&=0, & \eta y+\zeta''z&=1.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

The points of contact, being within each plane the intersection of the ray and the directrix, are easily obtained.

The rays within the three planes of coordinates which form one edge of a circumscribed parallelopiped meet the directrices within the planes forming the opposite edge.

## II.—*On Complexes of Luminous Rays within Biaxal Crystals.*

1. A single ray of light when meeting the surface of a doubly refracting crystal is divided into two rays determined by means of their four coordinates, $r$, $s$, $\varrho$, $\sigma$. All incident rays constituting a configuration, especially all rays starting from a luminous point and forming a conical surface, constitute within the crystal a new configuration, represented by the system of three equations between ray-coordinates. All incident rays constituting a congruency, emanating, for instance, in all directions from a luminous point, constitute within the crystal, after refraction, another congruency. Finally, a complex of incident rays, all rays, for instance, emanating in all directions from every point of a luminous curve, constitute within the crystal another complex of refracted rays. The congruency of refracted rays is represented by two, the complex by a single equation between ray-coordinates.

2. But before entering into the discussions indicated by the foregoing remarks, a short digression on double refraction might be desirable.

A biaxal crystal being cut along any plane whatever, we may suppose that this plane is congruent with $xy$, and that the point where an incident ray meets it is the origin of coordinates O. Let

$$x=pz, \quad y=qz \quad \ldots \ldots \ldots \ldots \tag{1}$$

be the equations of the incident ray, whence

$$\frac{x}{p}=\frac{y}{q}, \quad \ldots \ldots \ldots \ldots \tag{2}$$

the equation of the plane of incidence. In the moment of incidence the front of the corresponding elementary wave, perpendicular to the ray, will be represented by

$$z+qy+px=0. \quad \ldots \ldots \ldots \ldots \tag{3}$$

After the front of the wave has moved in air through the unit of distance, its equation becomes

$$z+qy+px=w \quad \ldots \ldots \ldots \ldots \tag{4}$$

on putting

$$1+p^2+q^2=w^2.$$

At this moment the front of the wave intersects $xy$ along a right line, which we may denote by RR, the equation of which is

$$qy+px=w. \quad \ldots \ldots \ldots \ldots \tag{5}$$

If the optical density of the surrounding medium increases, the value of $w$ decreases in the same ratio.

3. Around the point O, where the incident ray meets the section of the crystal, let the wave-surface be described as it is at that moment when the front of the elementary wave intersects $xy$ along RR. The position of the axes of elasticity of the crystallized medium being known with regard to the axes of coordinates, the equation of the wave-surface only depends upon three constants $a$, $b$, $c$, which are to be referred to the same

unit as $w$. If both systems of axes are congruent, the wave-surface is represented by the well-known equation

$$(a^2x^2+b^2y^2+c^2z^2)(x^2+y^2+z^2)-[a^2(b^2+c^2)x^2+b^2(a^2+c^2)y^2+c^2(a^2+b^2)z^2]+a^2b^2c^2=0, \quad (6)$$

which, for simplicity, may be written thus,

$$\Omega=0.$$

4. The wave-surface is intimately connected with three ellipsoids, the equations of which are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad (7)$$

$$a^2x^2+b^2y^2+c^2z^2=1, \quad (8)$$

$$\frac{x^2}{bc}+\frac{y^2}{ac}+\frac{z^2}{ab}=1. \quad (9)$$

By means of the *first* and the *second* ellipsoid the wave-surface may be obtained most easily. The third ellipsoid has been introduced by myself on account of the following remarkable property. With regard to this ellipsoid the wave-surface is its own polar surface, *i. e.* the polar plane of any point of the surface touches it in another point, and *vice versâ*, the pole of any plane tangent to the surface is one of its points.

The wave-surface and the three ellipsoids depend upon the same constants. When the crystal turns around the point of incidence O, both the surface and the three ellipsoids simultaneously turn with it. In the new position their equations involve three new constants, indicating the position of the axes of elasticity with regard to the axes of coordinates. Now the wave-surface may be represented by

$$\Omega'=0,$$

and the third ellipsoid in the corresponding position by

$$Ax^2+Bxy+Cy^2+2Dxz+2Eyz+Fz^2-1\equiv E=0. \quad (10)$$

From the six constants of this equation, which may be regarded as known, you may derive the six constants of the wave-surface by determining both the direction and the length of the axes of the third ellipsoid.

Within the plane $xy$, supposed to be any section whatever of the crystal, OX and OY may be directed along the axes of the ellipse along which this plane is intersected by the third ellipsoid. Accordingly the constant B disappears from the last equation. Besides, if OZ be directed along that diameter of the ellipsoid which is conjugate to the plane $xy$, and cease therefore, in the general case, to be perpendicular to it, both constants D and E likewise disappear.

5. According to HUYGHENS's principle, we obtain both rays into which an incident ray is divided, when entering the crystal, by the following general construction. Construct the two planes passing through the trace RR and tangent to the wave-surface described within the crystal around the point of incidence O. Let H and H' be the

points of contact within these planes. The two right lines OH, OH′ drawn through the point of incidence O and the two points of contact H, H′ will be the refracted rays.

By means of the theorem referred to in the last number I have replaced this construction by the following one, much easier to manage. Construct with regard to the third auxiliary ellipsoid E the polar line of the trace RR. This polar line, which may be denoted by SS, meets the wave-surface within the crystal in the two points H and H′, OH and OH′ being, as before, the two refracted rays.

The plane HOH′, containing both refracted rays OH, OH′, may be called *the plane of refraction.* There are, generally speaking, four tangent planes passing through RR, as there are four points where the wave-surface is intersected by SS. We get therefore four rays, all confined within the plane of refraction, but two of them, not entering the crystal, are foreign to the question.

6. The plane of refraction may be constructed solely by means of the third ellipsoid E. The details of this construction depend upon the well-known different modes of determining the polar line SS. On proceeding in this way we meet some remarkable corollaries concerning double refraction*.

7. The poles of all planes passing through the trace RR, represented by

$$qy+px=w \quad . \quad . \quad . \quad (5),$$

are points of SS. All right lines passing through the point of incidence O and these poles fall within the plane of refraction confining SS. These right lines may likewise be regarded as diameters of the ellipsoid E conjugate to diametral planes passing through the trace along which the surface of the crystal, *i. e.* the plane $xy$, is intersected by the wave-front in its primitive position, the trace being parallel to RR and represented by

$$qy+px=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence

*The plane of refraction is that diametral plane of the ellipsoid* E, *the conjugate diameter of which is perpendicular to the plane of incidence in* O.

* In concluding a former paper, "Discussion de la forme générale des ondes lumineuses" (Crelle's Journal, No. xix. pp. 1 & 91, Mai 1838), I gave the following construction:—

"Construisez, par rapport à l'ellipsoide directeur, la ligne droite polaire (SS) de celle qui est perpendiculaire au plan d'incidence en O′. Elle coupera la surface de l'onde, decrite autour du point O, en deux points. Les deux lignes droites qui vont du point O aboutir à ces points seront les deux rayons réfractés; tandis que les deux plans, qui, contenant la perpendiculaire en O′ (RR), passent par ces deux mêmes points seront les fronts des deux ondes planes correspondantes. Enfin il a été démontré, dans ce qui précède, que les deux plans de vibration sont ceux qu'on obtient en conduisant par les rayons lumineux (réfractés) des plans perpendiculaires aux fronts des ondes correspondantes."

At the present occasion I resume the discussion, announced by myself twenty-six years ago, of a part of this construction. More recently, in the eighteenth Leçon of his valuable work, 'Théorie mathématique de l'Elasticité' (1852), M. LAMÉ reproduces the curious relation between the wave-surface and the third ellipsoid. He presents in the following Leçon a remarkable theorem, "which is one of those immediately derived from this relation." [8]

Accordingly the plane of refraction, conjugate to (6), is represented by the equation

$$\frac{d\mathrm{E}}{dx}q=\frac{d\mathrm{E}}{dy}p, \quad \ldots \ldots \ldots \quad (12)$$

which may be expanded into the following one,

$$(\mathrm{A}x+\mathrm{B}y+\mathrm{D}z)q=(\mathrm{B}x+\mathrm{C}y+\mathrm{E}z)p, \quad \ldots \ldots \quad (13)$$

or

$$(\mathrm{A}q-\mathrm{B}p)x+(\mathrm{B}q-\mathrm{C}p)y+(\mathrm{D}q-\mathrm{E}p)z=0\,*. \quad \ldots \ldots \quad (14)$$

8. These equations remain unaltered if $p$ and $q$ vary in such a way that the ratio $\frac{p}{q}$ remains the same, *i. e.* if the angle of incidence vary while the plane of incidence remains the same. The same equations do not contain $w$, the value of which depends upon the density of the surrounding medium. Hence

*All rays of light confined within the same plane of incidence, after being divided into two by double refraction, are confined again within the same plane—the plane of refraction. This plane remains the same if the surrounding medium be changed.*

9. The plane $xy$, i. e. the surface of the crystal, containing the trace (11), its conjugate diameter, the equations of which are

$$\frac{d\mathrm{E}}{dx}=0, \quad \frac{d\mathrm{E}}{dy}=0, \quad \ldots \ldots \ldots \quad (15)$$

or

$$\left.\begin{aligned}\mathrm{A}x+\mathrm{B}y+\mathrm{D}z=0,\\ \mathrm{B}x+\mathrm{C}y+\mathrm{E}z=0,\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (16)$$

is confined within the plane of refraction, whatever may be the incident ray. The same may be proved analytically by observing that (12) is satisfied by means of the two equations (15). Hence

*A ray of light of any direction whatever meeting the surface of a biaxal crystal in a fixed point is so refracted that the plane containing both refracted rays passes through a fixed right line* (15).

* On representing any one of both refracted rays by the equations

$$x=rz, \qquad y=sz,$$

the last equation, written thus,

$$(\mathrm{A}q-\mathrm{B}p)r+(\mathrm{B}q-\mathrm{C}p)s+(\mathrm{D}q-\mathrm{E}p)=0, \quad \ldots \ldots \ldots \quad (1)$$

indicates a relation between the direction of the incident ray, determined by the constants $p$ and $q$, and the direction of the refracted one, determined by $r$ and $s$.

This equation will not be altered if the incident ray, moved parallel to itself, meet the section of the crystal in any point

$$x=\varrho, \qquad y=\sigma.$$

If $r$ and $s$ be regarded as variable, $\varrho$ and $\sigma$ being constant, the equation (1) represents the plane of refraction corresponding to the incident ray

$$x=pz+\varrho, \qquad y=qz+\sigma,$$

and containing both refracted rays.

*If without the crystal the plane of incidence turns round the perpendicular to the section, within the crystal the plane of refraction simultaneously turns round the diameter of the third ellipsoid conjugate to the section.*

10. In order to construct the plane of refraction, we want to know another diameter conjugate to any plane passing through the trace (11). In selecting among these planes the wave-front itself in its primitive position, the plane of refraction will be obtained by drawing a plane through both diameters conjugate to the section of the crystal and the primitive wave-front.

The wave-front in its primitive position is represented by

$$px+qy+z=0,$$

its conjugate diameter by the equations

$$\left.\begin{aligned}\frac{dE}{dx}&=p\cdot\frac{dE}{dz},\\ \frac{dE}{dy}&=q\cdot\frac{dE}{dz},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

which, if expanded, become

$$\left.\begin{aligned}Ax+By+Dz&=p(Dx+Ey+Fz),\\ Bx+Cy+Ez&=q(Dx+Ey+Fz).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In order to prove in the analytical way that the diameter conjugate to the primitive wave-front falls within the plane of refraction, it is sufficient to observe that, by eliminating $\frac{dE}{dz}$ between the two equations (17), the equation of the plane of refraction (12) is obtained.

11. If a ray of light meet the surface of a crystal in a given point, the third ellipsoid remains invariably the same as long as the position of the crystal is not altered. Therefore the diameter conjugate to the wave-front remaining likewise the same, whatever may be the section of the crystal passing through the point of incidence, the plane of refraction always passes through that fixed diameter. Again, if the incident ray, displaced parallel to itself, meet the surface of the crystal in a new point, this new point of incidence becomes the centre of the third ellipsoid, likewise displaced parallel to itself. The diameter conjugate to the primitive wave-front, always passing through the point of incidence, retains the same direction. We may finally observe that the surface of the crystal, if a curved one, may be replaced for any incident ray by the plane tangent to it in the point of incidence.

*If a ray of light meet a biaxal crystal in a given point, whatever may be the surface bounding the crystal and containing that point, the plane of refraction passes through a fixed right line.*

*If a system of parallel rays meet the surface of a biaxal crystal, each ray of which after double refraction is divided into two, there is within the crystal a fixed direction, not depending upon the shape of the surface, so that the directions of both refracted rays*

*into which any incident ray is divided, and that fixed direction, are confined within the same plane.*

12. By putting

$$Dq=Ep,$$

the equation of the plane of refraction becomes

$$(Aq-Bp)x+(Bq-Cp)y=0,$$

which, after eliminating $p$ and $q$, may be written thus,

$$(AE-DB)x+(BE-DC)y=0. \quad \ldots \ldots \ldots \quad (19)$$

In this case the plane of refraction is perpendicular to $xy$ and passes through OZ. The plane of incidence perpendicular to $xy$, or its trace within this plane, is represented by

$$Dy=Ex. \quad \ldots \ldots \ldots \quad (20)$$

It is easily seen that this trace is perpendicular to the trace of that diametral plane which, with regard to the ellipsoid E, is conjugate to OZ. Indeed this plane is represented by

$$\frac{dE}{dz}\equiv Dx+Ey+Fz=0,$$

and its trace within $xy$ by

$$Dx+Ey=0.$$

Each ray within the plane of incidence (20) is divided by double refraction into two, both confined within the same *vertical* plane of refraction. That is especially the case with regard to the ray incident at right angles; the corresponding plane of refraction, represented by (19), contains the incident ray and both the refracted rays.

13. Besides the vertical ray, there is in each plane of incidence one ray confined with both refracted rays within the same plane. After eliminating $p$ and $q$ between the general equations of the planes of incidence and of refraction,

$$qx=py,$$
$$(Ax+By+Dz)q=(Bx+Cy+Ez)p,$$

the following equation is obtained,

$$B(y^2-x^2)+(A-C)xy+(Dy-Ex)z=0, \quad \ldots \ldots \quad (21)$$

representing a cone of the second degree, the locus of incident rays which are confined within their corresponding planes of refraction. This cone passes through the vertical OZ, and intersects $xy$ within two right lines perpendicular to each other. These lines are congruent with the two axes of the ellipse

$$Ax^2+2Bx+Cy^2=1, \quad \ldots \ldots \ldots \quad (22)$$

along which the plane $xy$ is intersected by the ellipsoid E. (That is instantly seen by putting B=0 [4].) Hence both rays, grazing the surface of the crystal along the axes of the ellipse (22), are confined with both corresponding refracted rays within the same plane.

If especially the crystal be cut in such a way that $xy$ become a *circular* section of the ellipsoid E, each ray grazing the surface of the crystal will be contained within the corresponding plane of refraction. This plane therefore is easily obtained by means of the trace of the plane of incidence and the diameter OZ′ of the ellipsoid E conjugate to its circular section $xy$.

14. In the preceding numbers the plane of refraction has been determined without determining SS confined within it. This right line, passing through the infinitely distant pole of $xy$, is parallel to the diameter OZ′ conjugate to $xy$ and represented by the equations (16), which by eliminating successively $y$ and $x$ may be replaced by the following ones,

$$\left.\begin{aligned}(B^2-AC)x+(BE-CD)z=0,\\ (B^2-AC)y+(BD-AE)z=0.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (23)$$

The direction of SS being known, any one of its points, *i. e.* the pole of any plane passing through RR, will be sufficient to construct it. If the plane be parallel to the diameter just determined, its pole will fall within the plane $xy$, and may be also regarded as the pole of RR, with regard to the ellipse (22) along which this plane is intersected by E. The trace RR being represented by

$$qy+px=w,$$

where

$$w^2=1+p^2+q^2,$$

the two lines, the equations of which are

$$(Ax+By)\frac{w}{p}=1,$$

$$(Bx+Cy)\frac{w}{q}=1,$$

will meet in the pole mentioned. Hence, on denoting its coordinates by $x^0$ and $y^0$,

$$\left.\begin{aligned}x^0=\frac{Bq-Cp}{B^2-AC}\cdot\frac{1}{w},\\ y^0=\frac{Bp-Aq}{B^2-AC}\cdot\frac{1}{w}.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (24)$$

Finally, the equations of SS thus obtained are

$$\frac{x-x^0}{CD-BE}=\frac{y-y^0}{AE-BD}=\frac{z}{B^2-AC}. \quad \ldots\ldots\ldots\ldots \quad (25)$$

In order to complete the construction of the two refracted rays, the points (M, M′) in which SS meets the wave-surface $\Omega$ within the crystal are to be joined with O by means of two right lines OM and OM′.

15. If rays of every direction meet the crystal in O, the corresponding wave-fronts in that moment when, within the crystal, the wave-surface $\Omega$ is formed, will envelope a sphere,

$$x^2+y^2+z^2=1,$$

the radius of which is equal to unity. The locus of poles of the wave-fronts, if taken with regard to the ellipsoid E, is a new ellipsoid, which, referred to axes of coordinates directed along the axes of all auxiliary ellipsoids, is represented by the equation

$$\frac{x^2}{b^2c^2}+\frac{y^2}{a^2c^2}+\frac{z^2}{a^2b^2}=1,$$

or

$$a^2x^2+b^2y^2+c^2z^2=a^2b^2c^2. \quad \ldots\ldots\ldots\ldots \quad (26)$$

Its axes are obtained by multiplying the axes of the *second* auxiliary ellipsoid (8), to which it is similar, by *abc*.

16. The new fourth auxiliary ellipsoid (26) is fitted to connect the constructions of the refracted rays if, the section of the crystal remaining the same, the direction of the incident rays vary. Indeed a right line (MM′) drawn through any point Y of the fourth ellipsoid (26) parallel to OZ′, *i. e.* to the diameter conjugate to $xy$ with regard to the third ellipsoid E, meets the wave-surface $\Omega$, within the crystal, in two points M and M′. OM and OM′ will be the two refracted rays corresponding to that incident ray which is perpendicular to the plane conjugate to OY.

17. After this digression we resume our subject.

Let $xy$ be the section of a biaxal crystal and OZ perpendicular to it. Let a ray of any direction starting from any point of OZ meet the section of the crystal in a point the coordinates of which are

$$x-\varrho, \qquad y=\sigma.$$

Let

$$\left.\begin{aligned} x&=pz+\varrho,\\ y&=qz+\sigma \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (27)$$

be the equations of the incident ray. In order to express that this ray meets OZ we obtain the following relation,

$$\frac{p}{q}=\frac{\varrho}{\sigma}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (28)$$

Let

$$\left.\begin{aligned} x&=rz+\varrho,\\ y&=sz+\sigma \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (29)$$

be the equations of any one of the two corresponding refracted rays. Let us finally suppose that, without the crystal, $z$ is negative, within it, positive. Accordingly in the equations of the incident ray, positive values of $z$, in the equations of the refracted rays, negative ones are to be rejected.

Again, let

$$\Omega=0$$

be the general equation of the wave-surface, and

$$E\equiv Ax^2+2Bxy+Cy^2+2Dxz+2Eyz+Fz^2-1=0$$

the equation of the third auxiliary ellipsoid; the position of both being determined by the position of the crystal with regard to the axes of coordinates.

18. According to the footnote of [7], we have between the four constants $p$, $q$, $r$, $s$, upon which the direction of the incident and the refracted ray depends, the following relation,

$$(Aq-Bp)r+(Bq-Cp)s+(Dq-Ep)=0. \quad \ldots \ldots \quad (30)$$

By means of (28) this equation may be transformed into the following one,

$$(A\sigma-B\varrho)r+(B\sigma-C\varrho)s+(D\sigma-E\varrho)=0, \quad \ldots \ldots \quad (31)$$

and then represents a *complex of refracted rays*. As no supposition is made regarding the position of the luminous point on OZ, the corresponding incident rays may start in every direction from all its points. They constitute therefore a complex of rays emanating from OZ, perpendicular to the section of the crystal, and considered as a luminous right line. This complex of incident rays, after entering the crystal, passes into the complex of double refracted rays represented by the last equation.

19. By admitting that OX and OY, within the section of the crystal, were directed along the axes of the ellipse, along which $xy$ is intersected by the ellipsoid E, the constant B disappears from the equation of the complex, which then may be written thus,

$$(Ar+D)\sigma=(Cs+E)\varrho. \quad \ldots \ldots \quad (32)$$

We have hitherto supposed OZ to be perpendicular to $xy$, and will continue to do so for incident rays without the crystal; but for the refracted rays entering it (the axes OX, OY, perpendicular to each other, remaining the same) the direction of OZ may be changed by replacing it by the diameter OZ′ of the ellipsoid E, conjugate to $xy$. Then the constants D and E likewise disappear, and the equation of the complex assumes the most simple form,

$$Ar\sigma=Cs\varrho.$$

20. On denoting by $a_0$ and $b_0$ the two semiaxesof the ellipse along which $xy$ is intersected by the ellipsoid E, we get

$$A=\frac{1}{a_0^2}, \quad B=\frac{1}{b_0^2}.$$

We may suppose, too, that $a_0$ falling within OX, is greater than $b_0$ falling within OY, whence the square of the excentricity of the ellipse $e_0^2$ becomes $\frac{a_0^2-b_0^2}{a_0^2}$.

After having introduced the new constants, the last equation may be written in the following ways,

$$\frac{\sigma}{s}=\frac{a_0^2}{b_0^2}\cdot\frac{\varrho}{r}, \quad \ldots \ldots \quad (33)$$

$$\frac{r\sigma-s\varrho}{r}=e_0^2.\sigma, \quad \ldots \ldots \quad (34)$$

$$\frac{s\varrho-r\sigma}{s}=-\frac{a_0^2-b_0^2}{b_0^2}\cdot\varrho. \quad \ldots \ldots \quad (35)$$

Besides, on observing that $\frac{\sigma}{\varrho}=\frac{q}{p}$,

$$\frac{s}{r}=\frac{b_0^2}{a_0^2}\cdot\frac{q}{p} \quad \ldots \ldots \quad (36)$$

In order to get a geometrical interpretation of these equations, let any refracted ray of the complex be projected in the ordinary way on the three planes of coordinates XY, XZ and YZ; each axis of coordinates will be met by two of the three projections. The intercepts on OZ′ are $\frac{\sigma}{s}$ and $\frac{\varrho}{r}$; on OY, $\sigma$ and $\frac{r\sigma-s\varrho}{r}$; on OX, $\varrho$ and $\frac{s\varrho-r\sigma}{s}$. Hence

*With regard to all rays of the complex, the two intercepts on each axis of coordinates are in the same ratio.*

For OZ′, *i. e.* for the diameter of the ellipsoid E conjugate to the section of the crystal, this ratio is the ratio of the squares of the axes of the ellipse within this plane. For OY, *i. e.* for the shorter axis of this ellipse, it is equal to the square of its excentricity; for OX the greater axis equal to $\left(-e_0^2 \cdot \frac{a_0^2}{b_0^2}\right)$.

Finally, if any incident ray, without, be projected on the section $xy$ of the crystal along OZ, *i. e.* perpendicularly, and one of the two corresponding refracted rays, within the crystal, along OZ′, the projections thus obtained are the traces of the planes of incidence and of refraction, $\frac{q}{r}$ and $\frac{s}{r}$ indicating the trigonometrical tangents of the angles, between the two traces and the greater axis of the ellipse within the section $xy$. *The ratio of the tangents is equal to the ratio of the squares of the axes of the ellipse.*

21. In order to get a general idea of the distribution of the refracted rays constituting the complex, we may determine first the cone formed by rays passing through any given point within the crystal. If M be this point and $x_0$, $y_0$, $z'_0$ its coordinates, the equations

$$\left.\begin{aligned} x_0&=rz'_0+\varrho, \\ y_0&=sz'_0+\sigma, \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (37)$$

are to be combined with the equation of the complex, which, on putting $\frac{b_0}{a_0}=\beta$, may be written thus,

$$s\varrho=\beta^2 r\sigma. \quad \ldots \ldots \ldots \ldots \quad (38)$$

By eliminating $\varrho$ and $\sigma$, we get

$$x_0 s-\beta^2 y_0 r=(1-\beta^2)z'_0 rs. \quad \ldots \ldots \ldots \quad (39)$$

This equation shows that the locus of rays of the complex which pass through the point M is a cone of the second degree. Its equation in ordinary coordinates $x$, $y$, $z'$ ($z'$ being referred to OZ′) is

$$x_0(y-y_0)(z'-z'_0)-\beta^2 y_0(x-x_0)(z'-z'_0)=(1-\beta^2)z'_0(x-x_0)(y-y_0), \quad \ldots \quad (40)$$

From this equation we immediately derive that, whatever may be the position of M within the crystal, the cone always contains three rays parallel to OX, OY, OZ′, as well as a fourth ray passing through the origin O. Besides, the cone depends upon the only constant $\beta$, the ratio of the two axes of the ellipse, here represented by

$$\frac{x^2}{a_0^2}+\frac{y^2}{b_0^2}=1, \quad \ldots \ldots \ldots \ldots \quad (41)$$

along which $xy$ is intersected by the third auxiliary ellipsoid E.

The equation (39), only depending upon the ratio of the constants $x_0$, $y_0$, $z_0$, shows

that the cone in question of double refracted rays *is not at all altered if its centre moves along a right line passing through the origin* O.

22. In the peculiar case where M lies within the section of the crystal $xy$ all corresponding incident rays likewise meet in that same point, constituting the plane of incidence passing through OZ, and represented by

$$y'x=x'y.$$

Here the cone of refracted rays degenerates into a system of two planes, which after putting $z'_0=0$, are represented by

$$\left.\begin{array}{c} z'=0, \\ x_0(y-y_0)=\beta^2 y_0(x-x_0). \end{array}\right\} \quad \dots \dots \dots \quad (42)$$

The second of these equations represents the plane of refraction corresponding to the plane of incidence*.

23. If M fall within one of both the other planes of coordinates XZ and YZ, the cone of double refracted rays likewise degenerates into two planes.

24. Either by putting $z'=0$ in (40), or, after having eliminated $r$ and $s$ between the three equations (37) and (38), by replacing the remaining variables $\varrho$ and $\sigma$ by $x$ and $y$, we obtain

$$y_0x-\beta^2x^0y=(1-\beta^2)xy. \quad \dots \dots \dots \quad (43)$$

This equation represents, within $xy$, the trace of the cone of refracted rays which meet in M. It is an equilateral hyperbola, having its asymptotes parallel to OX and OY, and passing through the projection of M. The coordinates of its centre are

$$y=\frac{y_0}{1-\beta^2}\; x=-\frac{\beta^2x_0}{1-\beta^2},$$

whence

$$\frac{y}{x}=-\frac{1}{\beta^2}\frac{y_0}{x_0}.$$

As the equation (43) does not involve the constant $z'_0$, we conclude that

*The cone of double refracted rays continually changes if its centre be moved along a right line parallel to* OZ', *but its trace within the section of the crystal always remains the same hyperbola.*

25. Secondly, we may determine the curve enveloped by refracted rays confined within any given plane. If the plane be

$$tx+uy+vz+w=0,$$

* In the present researches, the auxiliary ellipsoid E, which may be considered as described round any point of the section of the crystal, as well as the wave-surface itself, has no other signification than to indicate by its constants the molecular constitution of the crystal so far as the transmission of luminous vibrations is concerned. Our equations only containing the ratio of these constants, the ellipsoid E and its elliptical trace (41) may be supposed here to have any dimensions whatever.

The last equation (42) represents the plane of refraction as it represents its trace within $xy$. It likewise represents, if the point M falls within the circumference of the ellipse (41), the normal to that curve in the point M. Hence is derived an elegant construction of the plane of refraction.

If within $xy$ round any point of incidence as centre the ellipse (41) be described, the traces of the planes, both of incidence and of refraction, are such two diameters of that ellipse, the second of which is parallel to the normal to it at the point where the first intersects it.

the equation of this curve will result from the combination of the equation of the complex

$$s\varrho=\beta^2 r\sigma \quad . \quad . \quad . \quad . \quad . \quad (38)$$

with the two equations

$$tr+us+v=0,$$

$$t\varrho+u\sigma+w=0,$$

expressing that a ray $(r, s, \varrho, \sigma)$ falls within that plane. By eliminating $r$ and $\varrho$, we obtain

$$ws-\beta^2 v\sigma+(1-\beta^2)us\sigma=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$\frac{1}{\sigma}$ and $\left(-\frac{s}{\sigma}\right)$ being the coordinates of the projection, within $xy'$, of the refracted ray. The projection envelopes an hyperbola; so does the ray itself within the given plane. The last equation (44) does not contain $t$, and therefore will not be altered if the given plane turns round its trace within YZ′, represented by

$$uy+vz'+w=0.. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Hence it follows that the projections of all refracted rays which meet that trace are tangents to the same hyperbola (44), the asymptotes of which are parallel to OY and OZ′, and which especially is touched by the trace itself, with regard to which

$$\sigma=-\frac{w}{u}, \quad \frac{\sigma}{s}=-\frac{w}{v}.$$

The refracted rays themselves are tangents to a hyperbolic cylinder having as base the hyperbola (44) and OX as axis.

26. In order to particularize, let us, in the first instance, suppose that the trace (45) is parallel to OZ′ and intersects OY in any point Q, OQ being equal to $\left(-\frac{w}{u}\right)$. Then $v$ being equal to zero, the equation (44) becomes

$$\big(w+(1-\beta^2)u\big)s=0,$$

indicating that the hyperbola of the general case degenerates into two points, falling within OY, one at an infinite distance, while the distance of the other (Q′) from O is

$$OQ'=\sigma=-\frac{1}{1-\beta^2}\frac{w}{u}=\frac{1}{1-\beta^2}OQ. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Accordingly the hyperbolic cylinder degenerates into two right lines, met by all refracted rays. One of the two lines within the plane $xy$ along which the crystal is cut is parallel to OX, and intersects OY in Q′, the other is infinitely distant. Hence all rays within a plane intersecting $xz'$ along a trace ($QZ'_0$) parallel to OZ′ are divided into two sets. The rays of one set being parallel to the plane $xy$ may be here omitted. The rays of the other set meet in a fixed point of that same plane along which the crystal is cut. If the plane turns round its trace $QZ'_0$ the fixed point moves, within $xy$, parallel to OX, describing a right line $Q'X_0$. Each ray meeting both right lines $QZ'_0$ and $Q'X_0$ is a ray of the complex.

27. If, in the second instance, the trace (45) is parallel to OY and intersects OZ′ in R, OR being equal to $\left(-\frac{w}{v}\right)$, the equation (44) becomes

$$ws=\beta^2 v\sigma,$$

representing a point of OZ′, the distance of which from O is

$$\text{OR}'=-\frac{\sigma}{s}=-\frac{1}{\beta^2}\frac{w}{v}=\frac{1}{\beta^2}\,\text{OR}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The hyperbolic cylinder therefore degenerates into a right line ($RX_0$) within $xz'$ parallel to OX and passing through R′. Hence

All refracted rays of the complex confined within a plane intersecting $yz'$ along a trace ($RY_0$) parallel to OY converge into a fixed point of the plane $xz'$. If the plane turns round its trace, that point describes, within $xz'$, a right line $RX_0$ parallel to OX. Each ray meeting both lines $RY_0$ and $R'X_0$ is a ray of the complex.

28. The axes of coordinates OX and OY may be interchanged by writing $a_0$ instead of $b_0$, and reciprocally. Then we get analogous results if, instead of traces within YZ′, we consider traces within XZ′. Especially we may immediately conclude from the last equation written thus,

$$b_0^2\,.\,\text{OR}'=a_0^2\,.\,\text{OR}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

that the relation between the two right lines $R'X_0$ and $RY_0$ is a mutual one.

29. All rays intersecting two fixed right lines constitute a *linear congruency*, the fixed right lines being its directrices (Sect. I., 55). Consequently *the complex of refracted rays may be generated in three different ways by a variable linear congruency.* In each case the two directrices of the congruency move parallel to any two of the three axes of coordinates OX, OY, OZ′, intersecting the third axis in two points, the distances of which from O are in a given ratio.

30. Hitherto we have supposed that the plane $xy$ is any section whatever of the crystal. Let us now, in particularizing again, admit that the crystal is cut along one of the two circular sections of the third auxiliary ellipsoid E, then represented by

$$\text{A}(x^2+y^2)+\text{F}z^2=1\,;$$

$\beta$ being equal to unity, the equation of the complex becomes

$$r\sigma=s\varrho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

In this peculiar case therefore all rays of the complex meet the diameter OZ′, conjugate with regard to E to its circular section $xy$. Hence *all refracted rays of the complex intersect* OZ′ *as all corresponding incident rays start from* OZ.

Both the diameter of the third auxiliary ellipsoid E perpendicular to its circular section $xy$, and its diameter conjugate to that section, fall within a principal section of the ellipsoid containing its greatest and least axis, and consequently also its two optic axes. The rectangular axes of coordinates OX and OY may, without changing the equation of the complex, turn round O within the section $xy$. If one of them, OX for instance, become

the vertical projection of OZ′, the plane $xz'$ is a *principal plane* of the ellipsoid E, containing the two optic axes, and OY the mean axis of the ellipsoid E.

31. If the plane $xy$ is a principal section of the third auxiliary ellipsoid E (and therefore of all auxiliary ellipsoids), the axis OZ′, becoming perpendicular to $xy$, is congruent with OZ. Then the equation of the ellipsoid E, referred to rectangular coordinates, becomes

$$\frac{x^2}{bc}+\frac{y^2}{ac}+\frac{z^2}{ab}=1,$$

and may be written thus,

$$ax^2+by^2+cz^2=abc.$$

Hence the equation of the complex is

$$ar\sigma=bs\varrho. \quad \ldots \quad (50)$$

If the crystal be turned round OY through an angle $\alpha$, we get, after replacing $x$ and $z$ by

$$x\cos\alpha-z\sin\alpha,$$

$$x\sin\alpha+z\cos\alpha,$$

the following equation of the ellipsoid E,

$$(a\cos^2\alpha+c\sin^2\alpha)x^2+by^2-2(a-c)\sin\alpha\cos\alpha\,.\,xz+(a\sin^2\alpha+c\cos^2\alpha)z^2=abc. \quad (51)$$

The axes of the elliptic trace within $xy$ being always directed along OY and OX, the equation of the complex assumes the form of the equation (32), which, after putting E=0 and

$$\mathrm{A}:\mathrm{C}:\mathrm{D}=(a\cos^2\alpha-c\sin^2\alpha):b:-(a-c)\sin\alpha\cos\alpha,$$

passes into the following one,

$$(a\cos^2\alpha-c\sin^2\alpha)r\sigma-bs\varrho-(a-c)\sin\alpha\cos\alpha\,.\,\sigma=0. \quad \ldots \quad (52)$$

32. The equations (51) and (52) of the last number belong to the case in which one of the three axes of elasticity, OY, falls within the section of the crystal. The two remaining axes of elasticity are confined within the plane XZ, where one of them, corresponding to C, makes with OZ an angle $\alpha$, this angle being counted towards OX.

The two equations may be regarded as representing the *general* case of *uniaxal crystals* cut along any plane whatever. Indeed let OC be the single optic axis making with the normal to the section $xy$ of the crystal any angle $\alpha$. Draw through OC the plane $xz$ perpendicular to $xy$, and OY perpendicular to that plane. The rectangular system of coordinates being thus determined, the equations (51) and (52), after having replaced $c$ by $a$, will belong to uniaxal crystals.

33. If the optic axis of an uniaxal crystal falls within the section $xy$, the equation of the complex, on putting $\alpha=\frac{1}{2}\pi$, becomes

$$cr\sigma=as\varrho.$$

In the case of uniaxal crystals, each plane passing through the optic axis may be regarded as a principal section of the ellipsoid E. Therefore the equation of the com-

plex assumes the form of the equation (50); the form of the two equations being the same as in the general case, where the direction of the third axis is oblique to $xy$.

If in the case of uniaxal crystals the circular section of E is congruent with the section $xy$ of the crystal, we get in order to represent the complex of double refracted rays, on putting $\alpha=0$, the following equation,

$$rs=s\varrho,$$

indicating that the plane of refraction is congruent with the plane of incidence, or, in other terms, that both the ordinary and the extraordinary ray into which any incident ray, starting from OZ, is divided by double refraction, likewise meet OZ.

34. The preceding fragmentary researches on double refraction—only calculated to present a new and curious instance of a complex—may be concluded by a last remark.

All the results we have hitherto obtained, especially the determination of the complex of double refracted rays, only depend, 1st, upon the direction of the diameter of the ellipsoid E conjugate to the section of the crystal; 2ndly, upon the ratio of the axes of the elliptical trace along which the same ellipsoid meets that section. Here, therefore, the third auxiliary ellipsoid E,

$$ax^2+by^2+cz^2=abc,$$

may be replaced by the following one,

$$ax^2+by^2+cz^2=1,$$

which is similar to it. It is immediately seen that, along the different directions, the reciprocal values of optical elasticity within the crystal are indicated by the *radii vectores* of the new ellipsoid, as the squares of these values are represented by the *radii vectores* of the second auxiliary ellipsoid,

$$a^2x^2+b^2y^2+c^2z^2=1.$$

## Additional Note.

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### I. *Coordinates of a right line.*

1. A right line, if considered as an axis round which a plane revolves, is determined by any two positions of the revolving plane; analytically, by means of two groups of plane-coordinates. If considered as a geometrical locus, described by a point, it is determined by any two positions of the moving point; analytically, by means of two groups of point-coordinates.

Let the plane- and point-coordinates

$$\frac{t}{w},\ \frac{u}{w},\ \frac{v}{w},\qquad \frac{x}{\varpi},\ \frac{y}{\varpi},\ \frac{z}{\varpi}$$

be such that

$$tx+uy+vz+w\varpi=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which equation, if geometrically interpreted, indicates that each point $\left(\frac{x}{\varpi}, \frac{y}{\varpi}, \frac{z}{\varpi}\right)$ falls within each plane $\left(\frac{t}{w}, \frac{u}{w}, \frac{v}{w}\right)$, or, which is the same, that each plane $\left(\frac{t}{w}, \frac{u}{w}, \frac{v}{w}\right)$ passes through each point $\left(\frac{x}{\varpi}, \frac{y}{\varpi}, \frac{z}{\varpi}\right)$. I called such coordinates "associated plane- and point-coordinates"*, and here we shall make use of that denomination. By two couples of associated either plane- or point-coordinates,

$$\frac{t}{w}, \frac{u}{w}, \frac{v}{w}, \qquad \frac{t'}{w'}, \frac{u'}{w'}, \frac{v'}{w'},$$

$$\frac{x}{\varpi}, \frac{y}{\varpi}, \frac{z}{\varpi}, \qquad \frac{x'}{\varpi'}, \frac{y'}{\varpi'}, \frac{z'}{\varpi'},$$

the *same* right line is determined.

We may employ homogeneous instead of ordinary equations†; accordingly each group of three coordinates is replaced by a group of four:

$$t, u, v, w, \qquad t', u', v', w',$$

$$x, y, z, \varpi, \qquad x', y', z', \varpi'.$$

2. Both planes $(t, u, v, w)$ and $(t', u', v', w')$, represented in point-coordinates by the equations

$$tx+uy+vz+w\varpi=0,$$

$$t'x+u'y+v'z+w'\varpi=0,$$

are arbitrarily chosen amongst those passing through the right line, and may be replaced by any two others, the equations of which have the form

$$(t+\mu t')x+(u+\mu u')y+(v+\mu v')z+(w+\mu w')\varpi=0,$$

where $\mu$ denotes any arbitrary coefficient. But the position of the right line with regard to the axes of coordinates OX, OY, OZ is not characteristically connected with such a plane, except in the case where the plane itself has a peculiar relation to the axes. There are four such cases: the plane may either pass through the origin, or project the right line on the three planes of coordinates. Accordingly, in putting

$$w+\mu w'=0, \quad v+\mu v'=0, \quad u+\mu u'=0, \quad t+\mu t'=0,$$

the last equation successively becomes

$$\left.\begin{aligned}(tw'-t'w)x+(uw'-u'w)y+(vw'-v'w)z&=0,\\(tv'-t'v)x+(uv'-u'v)y-(vw'-v'w)\varpi&=0,\\(tu'-t'u)x-(uv'-u'v)z-(uw'-u'w)\varpi&=0,\\-(tu'-t'u)y-(tv'-t'v)z-(tw'-t'w)\varpi&=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* Geometrie des Raumes, No. 5.

† I first introduced homogeneous equations into analytical geometry, CRELLE'S Journal, v. p. 1, 1830.

Any two of the four planes represented by these equations are sufficient to fix the position of the right line. They contain five constants, which by division may be reduced to four, the necessary number upon which the line depends. Besides the five constants in the two equations we meet a sixth one in both remaining equations. But the right line being determined by the former five, the sixth ought to be a function of them. The equation of condition, connecting the six constants, may, for instance, be obtained by adding the three last equations, after having multiplied the first of them by $-(tu'-t'u)$, the second by $(tv'-t'v)$, and the third by $-(uv'-u'v)$. Thus we obtain

$$(tu'-t'u)(vw'-v'w)-(tv'-t'v)(uw'-u'w)+(uv'-u'v)(tw'-t'w)=0. \quad . \quad . \quad . \quad (3)$$

The following six constants, taken with an arbitrary sign,

$$\pm(uv'-u'v), \quad \pm(tv'-t'v), \quad \pm(tu'-t'u), \quad \pm(tw'-t'w), \quad \pm(uw'-u'w), \quad \pm(vw'-v'w),$$

may be regarded *as the six coordinates of the right line.*

3. In quite a similar manner, when in order to fix the position of the right line we replace the two planes by the two points $(x, y, z, \varpi)$ and $(x', y', z', \varpi')$, we get the following equations in plane coordinates,

$$\left.\begin{aligned} (x\varpi'-x'\varpi)t+(y\varpi'-y'\varpi)u+(z\varpi'-z'\varpi)v&=0,\\ (xz'-x'z)t+(yz'-y'z)u-(z\varpi'-z'\varpi)w&=0,\\ (xy'-x'y)t-(yz'-y'z)v-(y\varpi'-y'\varpi)w&=0,\\ -(xy'-x'y)u-(xz'-x'z)v-(x\varpi'-x'\varpi)w&=0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

representing four points, the first of which is at an infinite distance on the right line of which the position is to be determined, while the three others are the points in which that line meets the three planes of coordinates. Accordingly we may likewise regard the six constants of the last four equations, taken with an arbitrary sign,

$$\pm(x\varpi'-x'\varpi), \quad \pm(y\varpi'-y'\varpi), \quad \pm(z\varpi'-z'\varpi), \quad \pm(yz'-y'z), \quad \pm(xz'-x'z), \quad \pm(xy'-x'y),$$

as *the six coordinates of the right line.* These six coordinates are connected by the following equation of condition:

$$(xy'-x'y)(z\varpi'-z'\varpi)-(xz'-x'z)(y\varpi'-y'\varpi)+(yz'-y'z)(x\varpi'-x'\varpi)=0. \quad . \quad . \quad (5)$$

4. In denoting the distance of the right line from the origin of coordinates by $\delta$, the angles with it makes with the three axes OX, OY, OZ by $\alpha$, $\beta$, $\gamma$, and the angles which the normal to the plane passing through it and the origin makes with the same axes by $\lambda$, $\mu$, $\nu$, the following relations are obtained:

$$\begin{array}{llllllllllll} \text{I.} & (uv'-u'v) & : & -(tv'-t'v) & : & (tu'-t'u) & : & (tw'-t'w) & : & (uw'-u'w) & : & (vw'-v'w) \\ \text{II.} & =(x\varpi'-x'\varpi) & : & (y\varpi'-y'\varpi) & : & (z\varpi'-z'\varpi) & : & (yz'-y'z) & : & -(xz'-x'z) & : & (xy'-x'y) \\ \text{III.} & = \cos\alpha & : & \cos\beta & : & \cos\gamma & : & \delta\cos\lambda & : & \delta\cos\mu & : & \delta\cos\nu. \end{array}$$

5. Hence we conclude that

$$\cos\alpha, \quad \cos\beta, \quad \cos\gamma, \quad \delta\cos\lambda, \quad \delta\cos\mu, \quad \delta\cos\nu$$

may likewise be regarded as line-coordinates. Here the equation of condition between the six coordinates becomes

$$\cos\alpha\cos\lambda+\cos\beta\cos\mu+\cos\gamma\cos\nu=0,$$

which, added to the two following ones,

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,$$
$$\cos^2\lambda+\cos^2\mu+\cos^2\nu=1,$$

reduces to four the number of constants upon which the position of the line depends.

6. The two sets of ratios I. and II. retain the same generality after putting $w=w'=\pm1$, $\varpi=\varpi'=\pm1$. If we suppose, again, that both planes and both points, by which the line is determined, are coincident, we get, choosing the under signs, two new sets of equal ratios,

$$\text{IV.}\quad =(udv-vdu):-(tdv-vdt):(tdu-udt):\quad dt\quad:\quad du\quad:\quad dv$$
$$\text{V.}\quad =\quad dx\quad:\quad dy\quad.\quad dz\quad:(ydz-zdy):-(xdz-zdx):(xdy-ydx).$$

Thus we obtain two systems of differential coordinates, $dx$, $dy$, $dz$ indicating the direction of the line, $dt$, $du$, $dv$ the direction of the normal to the plane passing through it and the origin of coordinates. We may regard $x$, $y$, $z$, $t$, $u$, $v$ as functions of time.

7. We can represent the direction of a *force* by the right line, and its intensity by the distance of the two points by which the position of the line is fixed. In denominating the projections of the force on OX, OY, OZ by X, Y, Z, and the projections of its moment with regard to the origin on YZ, XZ, XY by L, M, N, we obtain the following new set of equal ratios:

$$\text{VI.}\quad =X:Y:Z:L:M:N.$$

Therefore X, Y, Z, L, M, N may also be considered as six line-coordinates. The equation of condition between them becomes

$$XL+YM+ZN=0. \qquad (6)$$

8. The six coordinates of each system range into two groups of three, to each coordinate of one group corresponds one of the other. By exchanging the three axes of coordinates, the three couples of corresponding coordinates are exchanged, both groups remaining the same.

We may, in order to pass from the six coordinates of a right line to its five absolute coordinates, divide any five of them by the sixth. Here we meet two cases, in dividing either by a coordinate of the first or the second group.

9. Let us divide the first two and the three last terms of the ratios I. by the third $(tu'-t'u)$. In putting

$$\frac{uv'-u'v}{tu'-t'u}=r,\quad -\frac{tv'-t'v}{tu'-t'u}=s,\quad \frac{tw'-t'w}{tu'-t'u}=-\sigma,\quad \frac{uw'-u'w}{tu'-t'u}=\varrho,\quad \frac{vw'-v'w}{tu'-t'u}=\eta,$$

where, according to the equation of condition (3),

$$\eta=r\sigma-s\varrho,$$

$r$, $s$, $(-\sigma)$, $\varrho$, and $\eta$ will be the *five* absolute coordinates of the right line. The last two of the four equations (2), representing the planes projecting the right line on the planes XZ and YZ, as well as the projections themselves, may now be written thus,

$$x=rz+\varrho,$$
$$y=sz+\sigma,$$

$r$ and $s$ being the trigonometrical tangents of the angles made by the two projections with the axis OZ, $\varrho$ and $\sigma$ the segments intercepted by them on the axes OX and OY.

Again, let us divide the first five terms of the set of ratios II. by the sixth $(xy'-x'y)$. In putting

$$\frac{x\varpi'-x'\varpi}{xy'-x'y}=-\varkappa, \qquad \frac{y\varpi'-y'\varpi}{xy'-x'y}=\pi, \qquad \frac{z\varpi'-z'\varpi}{xy'-x'y}=\zeta,$$
$$\frac{yz'-y'z}{xy'-x'y}=p, \qquad -\frac{xz'-x'z}{xy'-x'y}=q,$$

where, according to the equation of condition (5),

$$\zeta=p\varkappa-q\pi,$$

$p$, $q$, $(-\varkappa)$, $\pi$, and $\zeta$ will be the *five* new coordinates. We meet four of them in the last two of the four equations (4), representing the two points where the planes XZ and YZ are intersected by the right line. These equations assume the following form,

$$t=pv+\pi w,$$
$$u=qv+\varkappa w,$$

and may, in denoting the coordinates of the points within their planes by $x_y$, $z_y$, and $y_x$, $z_x$, be written thus,

$$x_y t+z_y v+w=0,$$
$$y_x u+z_x v+w=0;$$

whence

$$p=-\frac{z_y}{x_y}, \quad \pi=-\frac{1}{x_y}, \quad q=-\frac{z_x}{y_x}, \quad \varkappa=-\frac{1}{y_x}.$$

We may add to the former six sets of equal ratios the two following:

$$\text{VII.} \quad = \quad r : s : \quad 1 \quad : (-\sigma) : \varrho : \eta(\equiv r\sigma-s\varrho)$$
$$\text{VIII.} \quad = -\varkappa : \pi : \zeta(\equiv p\varkappa-q\pi) : \quad p \quad : q : \quad 1.$$

10. We have thus obtained eight different systems of line-coordinates, the coordinates being the six terms of each of the eight sets of equal ratios I. to VIII. In changing the position of the origin and the direction of the axes of coordinates, the coordinates of each system are changed. But I do not here transcribe the formulæ of transformation of line-coordinates, observing only that these formulæ may be immediately transferred from one system to any other.

## II. *Complexes. Congruencies. Surfaces generated by a moving right line. Developable surfaces and curves of double curvature.*

11. A homogeneous equation between any six line-coordinates is said to represent the *complex* of those lines the coordinates of which verify that equation. According to the identity of ratios I. to VIII., the following equations,

$$F[(uv'-u'v),\ -(tv'-t'v),\ (tu'-t'u),\ (tw'-t'w),\ (uw'-u'w),\ (vw'-v'w)]=0,$$
$$F[(x\varpi'-x'\varpi),\ (y\varpi'-y'\varpi),\ (z\varpi'-z'\varpi),\ (yz'-y'z),\ -(xz'-x'z),\ (xy'-x'y)]=0,$$
$$F[\cos\alpha,\ \cos\beta,\ \cos\gamma,\ \delta\cos\lambda,\ \delta\cos\mu,\ \delta\cos\nu]=0,$$
$$F[(udv-vdu),\ -(tdv-vdt),\ (tdu-udt),\ dt,\ du,\ dv]=0,$$
$$F[dx,\ dy,\ dz,\ (ydz-zdy),\ -(xdz-zdx),\ (xdy-ydx)]=0,$$
$$F[X,\ Y,\ Z,\ L,\ M,\ N]=0,$$
$$F[r,\ s,\ 1,\ (-\sigma),\ \varrho,\ \eta]=0,$$
$$F[(-\varkappa),\ \pi,\ \zeta,\ p,\ q,\ 1]=0,$$

*represent the same complex*; F being supposed to indicate always the same homogeneous function of the different groups of line-coordinates. The *complex* is said to be of the $n$th *degree*, and represented by $\Omega_n$ if its equations are of that degree.

12. Starting from the first equation,

$$\Omega_n \equiv F[(uv'-u'v),\ -(tv'-t'v),\ (tu'-t'u),\ (tw'-t'w),\ (uw'-u'w),\ (vw'-v'w)]=0, \quad . \quad (1)$$

$t$, $u$, $v$, $w$ and $t'$, $u'$, $v'$, $w'$ are to be referred to any two planes passing through any line of the *complex*. Let one of the two planes ($t'$, $u'$, $v'$, $w'$) be any given one. Then the last equation, in regarding $t'$, $u'$, $v'$, $w'$ as constant and $t$, $u$, $v$, $w$ as variable, represents within the given plane a *curve* enveloped by tangent-planes ($t$, $u$, $v$, $w$). The lines of the *complex*, confined within the plane, also envelope the same curve, the class of which is the same as the degree of the *complex*. Hence

*A complex $\Omega_n$ of the* n*th degree being given, in each plane traversing space there is a curve of the* n*th class enveloped by lines of the complex.*

The equations of such curves fully agree with the general equation of the *complex* itself. We have only to consider in this equation $t'$, $u'$, $v'$, $w'$ as constant in referring them to the given plane, while $t$, $u$, $v$, $w$ are regarded as variable plane-coordinates.

If $n=1$, the curve in each plane is replaced by a point; each line within the plane passing through that point belongs to the linear complex.

If $n=2$, the curves enveloped are conics, which may degenerate into systems of two real or imaginary points.

13. If, in the second equation of the same *complex*,

$$\lambda\Omega_n \equiv F[(x-x'),\ (y-y'),\ (z-z'),\ (yz'-y'z),\ -(xz'-x'z),\ (xy'-x'y)]=0, \quad . \, . \quad (2)$$

where we put $\varpi'=\varpi=1$, and $\lambda$ denotes a constant, $x'$, $y'$, $z'$ are referred to any given

point in space and therefore regarded as constant, while $x$, $y$, $z$ are the variable coordinates of the points of any line of the *complex*, that equation represents a *cone* of the $n$th order, the geometrical locus of lines of the complex passing through the given point. Hence

*A complex of the* n*th degree being given, each point of space is the centre of a cone of the* n*th order into which lines of the complex converge.*

In linear complexes the lines meeting in a given point constitute a plane. If $n=2$, the cones are of the second order, and may degenerate into two real or imaginary planes.

14. The right lines constituting a complex $\Omega_n$ may be distributed either within planes traversing space, or according to points into which they converge. We hitherto considered $\Omega_n$ as a *complex of right lines*, the number of which is $\infty^3$. We may as well regard it either as a complex of curves, or as a complex of cones, the number both of curves and cones being $\infty^2$. Therefore we may say that

$$\Omega_n=0$$

*represents at the same time as well in each plane a curve of the* n*th class as cones of the* n*th order having each point of space as centre.*

The curve in a plane revolving round a given line, or moving parallel to itself, generates a surface. The cone the centre of which describes a given right line envelopes a surface. The number of surfaces both generated by the curve and enveloped by cones is $\infty$. There is one of each kind of surfaces corresponding to any given line, all surfaces will be exhausted if that line turns in all directions round any of its points. Accordingly we may likewise consider $\Omega_n$ as a complex of surfaces, either described by curves or enveloped by cones.

15. In denoting by $\mu$ any constant coefficient,

$$\Omega_n+\mu\Omega_m=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

represents an infinite number of complexes. The lines congruent in any two of them belong simultaneously to all. All these congruent lines constitute a *congruency* $(\Omega_n, \Omega_m)$, which we say is represented by the equations of the two complexes.

Each plane traversing space confines a curve of each of the two complexes, the $mn$ tangents common to both curves belong to the *congruency*. All curves within the same plane belonging to the different complexes (3) which pass through the congruency, touch the same $mn$ of its lines. Again, each point is the centre of a cone belonging to the different complexes (3). All such cones meet along the same $mn$ lines, likewise belonging to the *congruency*. Therefore in a congruency $(\Omega_n, \Omega_m)$ *there are* mn *lines confined within each plane as there are* mn *lines passing through each point.* The number of lines constituting a congruency is $\infty^2$.

If $m=1$, there are in each plane $n$ lines of the *congruency* $(\Omega_n, \Omega_1)$ passing through the same point, as $n$ of its lines converging into each point fall within the same plane; plane and point corresponding to each other.

16. In denoting by $\mu$ and $\nu$ any two constant coefficients,

$$\Omega \equiv \Omega' + \mu\Omega'' + \nu\Omega''' = 0 \quad \ldots \ldots \ldots \quad (4)$$

represents an infinite number ( $\infty^2$ ) of complexes. All these complexes meet along the lines which simultaneously belong to any three of them, especially to

$$\Omega' = 0, \quad \Omega'' = 0, \quad \Omega''' = 0. \quad \ldots \ldots \ldots \quad (5)$$

By means of these equations the position of such a line is determined, after having arbitrarily assumed the value of one of the four constants upon which the line depends; in other terms, three of these four constants are functions of the fourth, varying each by an infinitely small quantity if this one does. Hence we conclude that a line the coordinates of which verify the three equations (5), generates a *surface* in passing successively into all its positions. *This surface* $(\Omega', \Omega'', \Omega''')$ *is said to be represented by the system of the three equations* (5).

17. Any point of space being given, there are three cones described by lines which belong to the three complexes (5) and pass through the given point. Generally the three cones (11) do not intersect along the same line. In certain positions only of the point they do. In this case their common intersection belongs to the *surface* $(\Omega', \Omega'', \Omega''')$, and therefore the point itself also.

Put

$$\left.\begin{aligned} \lambda'\,\Omega' &\equiv F'\,[(x-x'), (y-y'), (z-z'), (yz'-y'z), -(xz'-x'z), (xy'-x'y)] = 0, \\ \lambda''\,\Omega'' &\equiv F''\,[(x-x'), (y-y'), (z-z'), (yz'-y'z), -(xz'-x'z), (xy'-x'y)] = 0, \\ \lambda'''\Omega''' &\equiv F'''[(x-x'), (y-y'), (z-z'), (yz'-y'z), -(xz'-x'z), (xy'-x'y)] = 0. \end{aligned}\right\} \quad . \quad (6)$$

If $x'$, $y'$, $z'$ are referred to any arbitrary point, and $x$, $y$, $z$ regarded as variable, these equations represent the three cones, $(x'y'z')$ being their common centre, and their generating lines belonging to the three complexes (5). Without changing the conditions of mutual intersection, the three cones may be moved parallel to themselves till the origin of coordinates becomes their common centre. After that displacement their equations are transformed into the following ones:

$$\left.\begin{aligned} F'\,[x, y, z, (yz'-y'z), -(xz'-x'z), (xy'-x'y)] &= 0, \\ F''\,[x, y, z, (yz'-y'z), -(xz'-x'z), (xy'-x'y)] &= 0, \\ F'''[x, y, z, (yz'-y'z), -(xz'-x'z), (xy'-x'y)] &= 0. \end{aligned}\right\} \quad \ldots \ldots \quad (7)$$

These equations being homogeneous with regard to $(x, y, z)$, will, in the general case, not be simultaneously verified by the three variables. In order to express that they subsist simultaneously, we obtain, after having eliminated $x$, $y$, $z$,

$$\varphi(x', y', z') = 0, \quad \ldots \ldots \ldots \ldots \quad (8)$$

$\varphi$ indicating a function which involves the primitive constants of the three complexes (5). This function might be rendered homogeneous by introducing $\varpi'$. This

equation, in regarding the coordinates as variable, represents in ordinary point-coordinates the *surface* which in line-coordinates is represented by the system of the three equations (5).

18. Likewise there are in each plane traversing space three curves enveloped by lines of the three complexes $\Omega'$, $\Omega''$, $\Omega'''$. In the general case these curves have no common tangent. In certain positions of the plane they have, and then the common tangent belongs to the *surface* $(\Omega', \Omega'', \Omega''')$. Reciprocally, within a plane passing through any generating line of the *surface*, the curves enveloped by the lines of any complex $\Omega$ touch the generating line, and continue to do so if the plane revolves round it. The plane in each of its positions is a *tangent-plane* of the *surface.*

Put

$$\left.\begin{aligned}\Omega' &\equiv F'\,[(uv'-u'v),\ -(tv'-t'v),\ (tu'-t'u),\ (t-t'),\ (u-u'),\ (v-v')]=0,\\ \Omega'' &\equiv F''[(uv'-u'v),\ -(tv'-t'v),\ (tu'-t'u),\ (t-t'),\ (u-u'),\ (v-v')]=0,\\ \Omega''' &\equiv F'''[(uv'-u'v),\ -(tv'-t'v),\ (tu'-t'u),\ (t-t'),\ (u-u'),\ (v-v')]=0.\end{aligned}\right\} \quad . \quad (9)$$

In regarding $t$, $u$, $v$ as variable plane-coordinates, and referring $t'$, $u'$, $v'$ to the traversing plane, these equations represent, within that plane, the three curves enveloped by lines of the three complexes $\Omega'$, $\Omega''$, $\Omega'''$. On this account they may be reduced to equations between two variables only, and therefore will not, in the general case, be verified by any values of the three variables reduced to two. By eliminating the variables between the last three equations, an equation,

$$\psi(t', u', v')=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

will be obtained, which, if $t'$, $u'$, $v'$ are regarded as variable, represents in plane-coordinates the *surface* $(\Omega', \Omega'', \Omega''')$.

19. In order to derive the equations (9) from the equations (6) (both systems of equations representing the same *surface*), we may first pass from (6) to the three new equations,

$$\begin{aligned}&F'\,[(yz'-y'z),\ -(xz'-x'z),\ (xy'-x'y),\ (x-x'),\ (y-y'),\ (z-z')]=0,\\ &F''[(yz'-y'z),\ -(xz'-x'z),\ (xy'-x'y),\ (x-x'),\ (y-y'),\ (z-z')]=0,\\ &F'''[(yz'-y'z),\ -(xz'-x'z),\ (xy'-x'y),\ (x-x'),\ (y-y'),\ (z-z')]=0,\end{aligned}$$

and then replace $x$, $y$, $z$, $x'$, $y'$, $z'$ by $t$, $u$, $v$, $t'$, $u'$, $v'$. The last equations are likewise obtained by merely exchanging amongst themselves the constant coefficients in each of the three equations (6). The way of exchanging is obvious. Hence, in considering that the equation (10) is derived exactly by the same algebraical operations from (9) as (8) from (7), we may conclude that (10) may be derived from (8) by a mere exchange of constants and a substitution of plane- for point-coordinates.

20. In a congruency $(\Omega_n, \Omega_m)$ there are $mn$ lines meeting in a given point. Two, three, four of these lines may coincide. In this case the cones of both complexes $\Omega_n$ and $\Omega_m$, the common centre of which is the given point, are tangent one to another, or osculate each other along the double or multiple line. In order to get the analy-

tical expression of these new conditions, we may, as we did before, replace both cones by such as have the origin as centre. In putting

$$\frac{x}{z}=p, \quad \frac{y}{z}=q,$$

the equations of these new cones may be written thus (No. 17),

$$\left.\begin{array}{l} f(p, q, x', y', z')=0, \\ f'(p, q, x', y', z')=0, \end{array}\right\} \quad \ldots \ldots \ldots \ldots \quad (11)$$

$f$ and $f'$ representing two functions of the variables $p$ and $q$, by means of which the lines constituting the two cones are determined, $x'$, $y'$, $z'$ being the coordinates of the given point. If two of the $mn$ intersecting lines of the two cones are coincident along any right line $(p, q)$, we get for the determination of that line, besides the two equations (11), the following new one,

$$\frac{df}{dq} : \frac{df}{dp} = \frac{df'}{dq} : \frac{df'}{dp},$$

which, if expanded, likewise assumes the form

$$f''(p, q, x', y', z')=0, \quad \ldots \ldots \ldots \ldots \quad (12)$$

$f''$ indicating a new function. By eliminating $p$ and $q$ between the three equations (11) and (12), we get an equation of the form

$$\psi(x', y', z')=0, \quad \ldots \ldots \ldots \ldots \quad (13)$$

representing, if $x'$, $y'$, $z'$ be regarded as variable, a *developable surface*, the locus of those points through which *double lines of the congruency* pass, or, in other terms, the locus of the double lines themselves.

In supposing that *three* intersecting lines of the two cones (11) fall within the same line $(p, q)$, the following new equation of condition is obtained

$$\frac{\frac{d^2f}{dp^2}\left(\frac{df}{dq}\right)^2-2\frac{d^2f}{dpdq}\cdot\frac{df}{dq}\cdot\frac{df}{dp}+\frac{d^2f}{dq^2}\left(\frac{df}{dp}\right)^2}{\frac{d^2f'}{dp^2}\left(\frac{df'}{dq}\right)^2-2\frac{d^2f'}{dpdq}\cdot\frac{df'}{dq}\cdot\frac{df'}{dp}+\frac{d^2f'}{dq^2}\left(\frac{df'}{dp}\right)^2}=\frac{\frac{df}{dp}}{\frac{df'}{dp}}=\frac{\frac{df}{dq}}{\frac{df'}{dq}},$$

which again may be expanded into an equation of the form

$$f'''(p, q, x', y', z')=0. \quad \ldots \ldots \ldots \ldots \quad (14)$$

This equation, combined with the three former equations (11) and (12), furnishes a new equation of condition,

$$\psi(x', y', z')=0. \quad \ldots \ldots \ldots \ldots \quad (15)$$

The system of the two equations (13) and (15) gives, as locus of points through which *triple* lines of the congruency pass, a curve of *double curvature*.

In pursuing the same course a new equation of the same form as (13) and (15) is

obtained, which, combined with these, indicates that there is a certain number of points into which quadruple lines of the congruency converge.

In congruencies of a peculiar description only we meet quintuple lines.

21. In quite the same manner we may determine the position of planes within which two, three, four of the $mn$ lines of the congruency $(\Omega_n, \Omega_m)$ coincide. In that case both curves within the plane, enveloped by lines of the complexes $\Omega_n$ and $\Omega_m$, touch or osculate one another on a common tangent.

In operating on the first two equations (9) as we did on the first two equations (6), we get, in order to represent in plane-coordinates the locus enveloped by planes confining a double line of the congruency, the following equation,

$$\psi(t, u, v)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which, as the remarks of No. 19 here likewise hold, is derived by a mere exchange of constants from (10). Each plane passing through a double line being an enveloping tangent plane of the represented surface, this surface degenerates into a *curve of double curvature*.

Another equation may be derived from (15) in the same way. Let it be

$$\psi'(t, u, v)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the system of the two equations (16) and (17) representing a *developable surface*, the tangent planes of which confine the triple lines of the congruency. Finally, there are certain tangent planes of the *developable surface* which confine the quadruple lines of the congruency. These planes, as well as the points of the curve of double curvature through which the quadruple lines pass, are determined by associated plane- and point-coordinates, both being functions of the constants of the congruency, and are obtained one from another by the above-mentioned exchange of these constants.

22. The double lines of a congruency constitute a *surface*, degenerated into a *developable* one, as they envelope a surface, degenerated into a *curve of double curvature*. The *developable surface* is represented in point-coordinates by a single equation (13), in plane-coordinates by the system of two equations (16) and (17). *The curve of double curvature* is represented in plane-coordinates by a single equation (16), in point-coordinates by the system of two equations (13) and (15). The tangent-planes of the *surface*, confining triple lines of the congruency, osculate the curve; the points of the *curve*, through which these triple lines pass, are osculating points of the *surface*, in which three consecutive tangent planes meet. The *curve*, in certain points where the tangent is an osculating one, is osculated by a plane in four points. Through such a point pass four consecutive tangent planes of the *surface*, the common intersection of which is a line of inflexion of the developable *surface*. The quadruple lines of the congruency pass through such points, and are confined within such planes*.

* In two remarkable papers "On a New Analytical Representation of Curves in Space," published in the third and fifth volume of the Quarterly Journal of Mathematics, Professor CAYLEY employed before me, in order to represent cones, the six coordinates of a right line, depending upon any two of its points. Having lately

### III. *On a new System of Coordinates.*

23. We have hitherto determined the position of a right line in space in making use of the ordinary system of three axes OX, OY, OZ intersecting each other. The new question is whether we may substitute for this system another, by means of which we are enabled to fix immediately the position of a right line without recurring to points and planes.

In the ordinary system of coordinates, (1) the position of a point is determined by means of three planes parallel to the planes of coordinates and meeting in that point, (2) the position of a plane by a linear equation between the three coordinates of a point, regarded as variable; both point and plane depending upon three constants.

In an analogous way a right line is determined by the intersection of four linear complexes. Such a linear complex depends upon the position of its axis and a constant (paper presented, No. 29). A right line, regarded as the direction of a *force*, belongs to the complex, if the moment of rotation of the force with regard to the axis, divided by its projection on the axis, be equal to the constant. Accordingly any four axes in space being given, the position of a right line is fixed by means of four constants, obtained by dividing the four moments of rotation with regard to the four axes by the four corresponding projections on the same axes.

The four axes of the complexes constitute the new system of coordinates; the four constants are the four coordinates of the given right line. The right line intersecting the four axes is the origin of coordinates, its four coordinates being equal to zero.

In the new system of coordinates a right line is determined in the most general way by its four coordinates; but an equation between the four coordinates is not in a general way sufficient to represent a linear complex, depending as it does on five constants.

We may *ad libitum* increase the number of coordinates of a right line.

24. Let P, Q, R, S, T, U . . be the axes of any number of complexes, and $p, q, r, s, t, u$ . . the corresponding coordinates of a given right line (according to the last number). Let

$$\Omega_p \equiv \Xi_p - p = 0, \quad \Omega_q \equiv \Xi_q - q = 0, \quad \Omega_r \equiv \Xi_r - r = 0,$$
$$\Omega_s \equiv \Xi_s - s = 0, \quad \Omega_t \equiv \Xi_t - t = 0, \quad \Omega_u \equiv \Xi_u - u = 0 \ldots$$

be the equations of the complexes. In order to express that the complexes meet along the same line, the following equations of condition are obtained,

$$\left.\begin{aligned} \Omega_t &\equiv \varkappa\Omega_p + \lambda\,\Omega_q + \mu\,\Omega_r + \nu\,\Omega_s, \\ \Omega_u &\equiv \varkappa'\Omega_p + \lambda'\Omega_q + \mu'\Omega_r + \nu'\Omega_s, \end{aligned}\right\} \qquad (18)$$
$$\cdots\cdots\cdots$$

only seen the papers, I hasten to mention it now. But, besides the coincidence referred to, the leading views of Professor CAYLEY's paper and mine have nothing in common. On this occasion I may state that the principles upon which my paper is based were advanced by me, nearly twenty years ago (Geometry of Space, No. 258), but this had entirely escaped from my memory when I recurred to Geometry some time since.

where we may suppose that P, Q, R, S are the former four axes of coordinates; $\varkappa$, $\varkappa'$, $\lambda$, $\lambda'$, $\mu$, $\mu'$, $\nu$, $\nu'$ indicate any constant coefficients.

In putting the coordinates $p$, $q$, $r$, $s$, $t$, $u$.. equal to zero, the general equations of the complexes become

$$\Xi_p=0,\quad \Xi_q=0,\quad \Xi_r=0,\quad \Xi_s=0,\quad \Xi_t=0,\quad \Xi_u=0.$$

These new equations represent complexes of a peculiar kind, the lines of which intersect their axes; they may be said to represent the axes themselves.

In order to satisfy the equation (18), we put

$$\left.\begin{aligned} \Xi_t &\equiv \varkappa\,\Xi_p+\lambda\,\Xi_q+\mu\,\Xi_r+\nu\,\Xi_s, \\ \Xi_u &\equiv \varkappa'\Xi_p+\lambda'\Xi_q+\mu'\Xi_r+\nu'\Xi_s, \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (19)$$
$$\ldots\ldots\ldots\ldots$$

whence

$$\left.\begin{aligned} t &=\varkappa p+\lambda q+\mu r+\nu s, \\ u &=\varkappa' p+\lambda' q+\mu' r+\nu' s. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (20)$$

The equations (19) require that the *origin* met by the axes P, Q, R, S be likewise met by the new axes T, U...

Therefore $p$, $q$, $r$, $s$, $t$, $u$.. may be regarded as coordinates of the right line along which all complexes meet; the axes of the complexes intersecting the same right line being the axes of coordinates. A right line being completely determined by the first four coordinates, those remaining depend upon them by linear equations (20).

The system of four axes of coordinates depends upon 16, of five axes upon 19, of six upon 22 constants.

Having thus established a system of coordinates which, independently of points and planes, fixes the position of a right line in space, we are enabled, by regarding right lines as elements of space, to reconstruct the whole geometry without recurring to the ordinary system. Here we are guided by analogy. As far as I may judge, the task is a most grateful but at the same time a long and laborious one.

## IV. *Geometry of Forces.*

25. In recapitulating the contents of the first three paragraphs of this note, new considerations have been suggested to me, which seem calculated, while greatly increasing again this kind of inquiry, to put the key-stone to it. Hitherto, when I borrowed technical terms from mechanical science, the only intention was to simplify the expression. But *force* may be regarded as a merely geometrical notion, and there is only one step more to be taken in order to arrive at a "*Geometry of Forces*," as there is a geometry based on the notion of right lines.

Forces depend upon five independent constants, four of which indicate their position, while the fifth indicates their intensity. We may call these constants the *five coordinates of the forces.*

In order to fix the direction of a *force*, we may employ line-coordinates and choose the following,

$$X, Y, Z, L, M, N,$$

indicating the projections of the *force* on the three axes of coordinates OX, OY, OZ, and its three moments of rotation with regard to these axes. Between them the following equation of condition holds good,

$$XL+YM+ZN=0$$

(see No. 7). The quotients obtained by dividing any five of them by the sixth are the absolute values of coordinates. From these quotients the intensity of the force has disappeared.

The *same six constants*, reduced by the last equation to five independent ones, *may be regarded as the absolute values of the coordinates of the force*. Instead of homogeneous equations between them, if regarded as variable, representing complexes of lines (of directions of the forces), we now get ordinary equations between the same variables representing *complexes of forces*.

The extension of all former developments thus indicated immediately occurs to us. A single instance may be referred to here. *Forces* constituting a linear complex are such passing in all directions through each point of space as have their intensity equal to the segments taken on their directions from the point to a certain plane corresponding to it. Forces common to two linear complexes and passing through a given point are confined within the same plane, the distance from the points where their directions meet a given line within the plane being their intensity. Forces, the coordinates of which verify simultaneously three linear equations, are distributed through space in such a manner that there is one force of a given intensity passing through each point of space.

The general contents of this note (except § IV.) were in a verbal communication presented by me at the last Birmingham Meeting of the British Association. As they concern the principles on which the original paper is based, giving to them a symmetry and a generality I was not before aware of, I thought it necessary to add the note to that paper. At the same time I also endeavoured to give an idea of the great fertility of the method developed. But as I am now preparing a volume for publication on this subject, I do not think it suitable to enter here into any details. The work will embrace the theory of the general equation of the second degree between line-coordinates, requiring no means of discussion but those employed by me in the case of equations of the same degree between point- or plane-coordinates. The complex of lines represented by such an equation may be regarded likewise as a complex of *curves of the second class*, one of which is confined in each plane, or as a complex of *cones of the second order*, each point of space being the centre of such a cone. In reducing the number of constants upon which the complex depends from 19 to 9, we pass in parti-

cularizing step by step from the general complex to a surface of the second order and class, determined by its tangents.

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I intend resuming the consideration of the mechanical part of this note. Then a last generalization will occur to us, the equation of condition, hitherto admitted between the six coordinates $x$, $y$, $z$, L, M, N, being removed.

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### I. *On Linear Complexes of Right Lines.*

complex may be regarded as two congruent conjugate lines. Principle of polar reciprocity applied, 20. Construction of the plane corresponding to a given point, of the point corresponding to a given plane, 21, 22. Geometrical determination of the constant of the general equation of the complex. There is a characteristic direction given by the double equation $\frac{x}{D}=\frac{y}{E}=\frac{z}{F}$. If that direction falls within $xy$, the term $(s\varrho-r\sigma)$ disappears and the general equation becomes linear. If any plane perpendicular to it is taken as one of the three planes of rectangular coordinates, and the corresponding point within it as origin, the general equation assumes one of the forms, $s=k\varrho$, $r=k\sigma$, $s\varrho-r\sigma=k$. A linear complex may, without being altered, turn round a fixed line, and move along it parallel to itself, 23–29. Geometrical interpretation of the last equations, 30. Points and planes corresponding to one another with regard to the complex $s\varrho-r\sigma=k$. Geometrical interpretations, 31, 32. Generalization, 33. Conjugate lines with regard to the complex $s\varrho-r\sigma=k$, 34. A linear complex depends upon five constants, four of which give the position of its axis, 35. Formulæ of the transformation of ray-coordinates corresponding to any displacement of the axes of coordinates, 36–38. Analytical determination of the axis of a complex, represented by the general equation. Determination of $k$, 39–43. In the peculiar case in which $k$ is equal to zero, all rays meet the axis of the complex, 44. Rays passing through the same point, 45.

*Linear congruencies of rays.*—A linear congruency, along which an infinite number of complexes intersect each other, is represented by the equations of any two of them. Through a given point of space only one ray passes, corresponding to that point, as there is only one ray confined within a given plane, 46. There is in each complex passing through the congruency one line conjugate to a given right line: all these lines belong to one generation of a hyperboloid, the second generation of which contains rays of the congruency. Generation of a linear congruency by a variable hyperboloid, 47–49. Characteristic section of a congruency to which the axes of all passing complexes are parallel. The axis of the congruency is a fixed right line, perpendicular to that section on which the axes of all complexes meet at right angles, 50, 51. The locus of points having in all complexes the same corresponding plane is a system of two right lines, the directrices of the congruency. Central plane parallel to both directrices and equidistant from them. The directrices may be real or imaginary, 52–54. In the first case there are amongst the complexes two of a peculiar description [44] having both directrices as axes. All rays of the congruency meet both its directrices, 55. The peculiar case in which one of the two directrices is infinitely distant, 56. Each of any two complexes being given by means of its constant $k$ and the position of its axis, to determine both directrices of the congruency, 57–59. A congruency being given by means of its two directrices, to determine the constants and the axes of the complexes passing through it. Centre of the congruency. The two secondary axes within the central plane, 60. Locus of the axes of all complexes meeting along the same congruency, 61.

*Linear configurations of rays* represented by the equations of three linear complexes.

An infinite number of congruencies meet along a linear configuration. Generally it is a hyperboloid. Its rays constitute one of its generations, while the directrices of all congruencies constitute the other, 62. The central planes of all congruencies meet in the same point: the centre of the configuration. Its diameters meet both directrices of the different congruencies, 63. A configuration is determined by means of three complexes, or by means of three congruencies, obtained by combining them two by two. Three couples of planes drawn through both directrices of each of the congruencies parallel to its central plane constitute a parallelopiped circumscribed to the hyperboloid. Each ray intersects all six directrices. The ray within each of the six planes is parallel to the directrix within the opposite plane; the point in which it meets the directrix within the same plane is the point of contact. Three diameters determined by both points of contact within the three couples of opposite planes. Imaginary diameters correspond to imaginary directrices, asymptotes to congruent directrices, 64. A hyperboloid being given, we may return to the congruencies and complexes which constitute it, 65. The equations of the configurations transformed into an equation between $x$, $y$, $z$, 66.

### II. *On Complexes of Luminous Rays within Biaxal Crystals.*

Complexes of doubly refracted rays corresponding to complexes of incident rays, 1.

*Digression on double refraction.* HUYGHENS'S principle. The author's construction presented, 1838. Auxiliary ellipsoids. The ellipsoid E, with regard to which the wave-surface is its own polar surface. The plane of refraction, containing both refracted rays, passes through SS, the polar line of RR, along which the surface of the crystal is intersected by the front of the incident elementary wave at that moment when, within the crystal, the wave-surface is formed, 2–6. The plane of refraction is congruent with the diametral plane of E, the conjugate diameter of which is perpendicular to the plane of incidence in O, 7. All rays incident within the same plane are, after double refraction, confined again within the same plane, 8. While the plane of incidence turns round the vertical in O, the corresponding plane of refraction turns round that diameter of E, the conjugate diametral plane of which is the surface of the crystal, 9, 10. Whatever may be the plane or curved surface met by an incident ray in any given point O, all corresponding planes of refraction pass through a fixed right line, 11. Peculiar cases of complexes. The plane of refraction perpendicular to the surface of the crystal. The incident and the two refracted rays confined within the same plane. A circular section of E falling within the surface of the crystal, 12, 13. Analytical determination of SS, 14. A fourth auxiliary ellipsoid, 15, 16.

*Complex of doubly refracted rays* determined by means of E. Its equation depending upon the constants of E, 17, 18. By taking as axes of coordinates three conjugate diameters of E, two of which, falling within the surface of the crystal, are perpendicular to each other, the general equation of the complex becomes $r\sigma = ks\varrho$, the constant $k$ being the ratio of the squares of the two rectangular diameters, 19. Geometrical interpretation, 20. Refracted rays of the complex passing through a given point consti-

tute a cone of the second order. The cone remains the same if the given point moves along a right line, passing through O, 21. Peculiar cases, 22–24. A hyperbola enveloped by the doubly refracted rays within any given plane. Its determination, 25. Peculiar cases. Geometrical interpretations, 26–28. The complex generated in three different ways by a variable linear congruency, 29. Peculiar case of a complex, the crystal being cut along a circular section of E. All doubly refracted rays meet that diameter of E the conjugate plane of which is the circular section, 30. Peculiar case in which the surface of the crystal is a principal section, 31. Case of uniaxal crystals, 32, 33. The ellipsoid E replaced by a new ellipsoid, the *radii vectores* of which indicate the reciprocal values of optical elasticity, 34.

*Additional Note.*

Coordinates of a right line, 1–10. Complexes. Congruencies. Surfaces generated by a moving right line. Developable surfaces and curves of double curvature, 11–22. A new system of coordinates, 23, 24. Geometry of forces, 25.